Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 10 Dr. M. Zaal
Optional – please hand in for correction in class on 10.07.2013

Problem 1 (Entropy/flux pairs)
Consider the shallow water equations and the barotropic Euler equations.

1. Suppose that Φ is an entropy for the shallow water equations

$$\begin{cases} \phi_t + (v\phi)_x = 0, \\ v_t + \left(\dfrac{v^2}{2} + \phi\right)_x = 0. \end{cases}$$

   Prove

$$\frac{\partial^2 \Phi}{\partial v^2} = \phi \frac{\partial^2 \Phi}{\partial \phi^2}.$$

2. Show that $\Phi = \rho v^2/2 + P(\rho)$ is an entropy for the barotropic Euler equations

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p)_x = 0, \end{cases}$$

   provided that $P''(\rho) = p'(\rho)/\rho$, $\rho > 0$. Confirm that Φ is convex in the proper variables. What is the corresponding entropy flux?

Problem 2 (Shocks in scalar conservation laws)
Assume that u is an entropy solution of the scalar conservation law $u_t + F(u)_x = 0$, and that u is smooth on either side of a curve $\{x = s(t)\}$.

1. Prove that along this curve the left and right hand limits of u satisfy the relations

$$F(z) \geq \frac{F(u_r) - F(u_l)}{u_r - u_l}(z - u_r) + F(u_r)$$

   if $u_l \leq z \leq u_r$, and

$$F(z) \leq \frac{F(u_r) - F(u_l)}{u_r - u_l}(z - u_r) + F(u_r)$$

   if $u_r \leq z \leq u_r$. This is called *condition E*.

2. What does condition E imply if F is uniformly convex?

Problem 3 (Travelling waves in semilinear diffusion)
Consider the semilinear diffusion equation

$$u_t = \Delta u + f(u) \tag{1}$$

where f is such that $f(0) = f(a) = f(1) = 0$ for some $a \in (0, 1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(z)\,\mathrm{d}z > 0. \tag{2}$$

Note that $u \equiv 0$, $u \equiv 1$ and $u \equiv a$ are stationary solutions of (1).

1. Linearize (1) around the three given stationary solutions, that is, find linear operators L_0, L_1 and L_a such that $(u-u_0)_t = L_{u_0}(u-u_0) + o(\|u-u_0\|)$ for u near u_0 in some appropriate norm. Conclude that 0 and 1 are stable and a is unstable.

The goal is to find a travelling wave solution connecting 0 and 1, that is, a profile $v : \mathbb{R} \to \mathbb{R}$ and speed σ such that $\lim_{s\to-\infty} v(s) = 0$, $\lim_{s\to+\infty} v(s) = 1$, $\lim_{s\to\pm\infty} \dot{v}(s) = 0$ and $u(x,t) = v(x-\sigma t)$ solves (1).

2. Consider the system

$$\begin{cases} \dot{v} = w, \\ \dot{w} = -\sigma w - f(v), \end{cases} \tag{3}$$

where $\sigma < 0$. Show that $(0,0)$ and $(1,0)$ are steady states of this system. Linearize the system around the steady states, and find the stable and unstable eigenvectors.

3. Define the *stable and unstable manifolds* of the equilibria (3) as

$$W_s(v_0,w_0) := \left\{ (v(0),w(0)) \in \mathbb{R}^2 : \lim_{s\to+\infty} (v(s),w(s)) \to (v_0,w_0) \right\},$$
$$W_u(v_0,w_0) := \left\{ (v(0),w(0)) \in \mathbb{R}^2 : \lim_{s\to-\infty} (v(s),w(s)) \to (v_0,w_0) \right\}.$$

How are the stable / unstable eigenvectors related to the stable / unstable manifold?

4. Suppose that $W_u(v_0,w_0) \cap W_s(\tilde{v}_0,\tilde{w}_0) \neq \emptyset$ for some equilibria $(v_0,w_0), (\tilde{v}_0,\tilde{w}_0)$. Show that there exists a solution $(v(s),w(s))$ of (3) such that

$$\lim_{s\to-\infty} (v(s),w(s)) = (v_0,w_0), \qquad \lim_{s\to+\infty} (v(s),w(s)) = (\tilde{v}_0,\tilde{w}_0),$$
$$\lim_{s\to\pm\infty} (\dot{v}(s),\dot{w}(s)) = 0.$$

5. Suppose that $W_u(0,0) \cap W_s(1,0)$ is nonempty for some $\sigma < 0$ (this can be shown using (2)). Show that there exists a travelling wave solution of (1) connecting 0 and 1.
