Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 9 Dr. M. Zaal
Due in class on 03.07.2013

Problem 1 (Waves in shallow water – 8 MARKS)
Consider the one-dimensional shall-water equations

$$\begin{cases} \phi_t + (v\phi)_x = 0, \\ v_t + \left(\dfrac{v^2}{2} + \phi\right)_x = 0, \end{cases} \tag{1}$$

which is a system of conservation laws with flux function $\mathbf{F}(z) = (z_1 z_2, \frac{z_2^2}{2} + z_1)$.

1. Find the eigenvalues $\lambda_k(z)$ and $\mathbf{r}_k(z)$ for $z \in \mathcal{S} := (0, +\infty) \times \mathbb{R}$, and show that DF is strictly hyperbolic in $\mathcal{S}$. Is any of the pairs $(\lambda_k, \mathbf{r}_k)$ genuinely nonlinear or linearly degenerated?

2. For $z \in \mathcal{S}$, find the rarefaction curves $R_k(z)$, and split $R_k(z)$ into $R_k^+(z)$ and $R_k^-(z)$ as defined in Evans, §11.2.

3. For $\eta > 0$, find the shock set $S(\eta, 0)$, and split $S(\eta, 0)$ into $S_k^{\pm}(\eta, 0)$ as defined in Evans, §11.2.

4. Set $\xi := 7 + 2\sqrt{6} = (1 + \sqrt{6})^2$, and find $R_1^+(\xi, 0) \cap S_2^+(2, 0)$. *Hint*: in order to simplify the computations, it may be convenient to parametrize the rarefaction and shock curves by their second coordinate. At some point, you may wish to solve an equation numerically.

5. Use the results to construct an integral solution of (1) with initial conditions

$$v(0, x) = 0, \qquad \phi(0, x) = \begin{cases} \xi, & \text{if } x < 0, \\ 2, & \text{if } x > 0. \end{cases}$$

6. Sketch the (x, t) half plane and indicate where ϕ and v are constant. Sketch the graph of $\phi(1, x)$, and use arrows to indicate the water flow.

Problem 2 (Riemann invariants for shallow water)
Consider the one-dimensional shall-water equations (1).

1. Rewrite the equations in the form

$$\phi_t + \lambda_i \phi_x \pm a_i(\phi, v)\,(v_t + \lambda_i v_x) = 0$$

2. Define x_i by

$$\dot{x}_i(t) = \lambda_{2-i}(\phi(x_i(t), t), v(x_i(t), t)), \qquad x(0) = x_0.$$

   Rewrite into differential equations involving the derivatives of ϕ and v along x_i, and use the result to find Riemann invariants w^i.

3. Suppose that ϕ_0 and v_0 are smooth and compactly supported, and that $\phi_0(x) > -1$ for all $x \in \mathbb{R}$. Show that there cannot exist a smooth positive solution of (1) with initial condition

$$\phi(x,0) = 1 + \phi_0(x), \qquad v(x,0) = v_0(x)$$

if

$$\frac{\partial v_0}{\partial x} < 2 \left| \frac{\partial}{\partial x} \sqrt{1+\phi_0} \right|$$

for any $x \in \mathbb{R}$. Can you interpret this requirement physically?
