Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 8 Dr. M. Helmers, Dr. M. Zaal
Due in class on 26.06.2013

Problem 1 (Behaviour of eigenvectors)
For $z \in \mathbb{R}$, $z \neq 0$ define the matrix function

$$B(z) = e^{-1/z^2} \begin{pmatrix} \cos(2/z) & \sin(2/z) \\ \sin(2/z) & -\cos(2/z) \end{pmatrix},$$

and set $B(0) = 0$. Show that B is C^∞ and has real eigenvalues, but we cannot find unit-length right eigenvectors $\{r_1(z), r_2(z)\}$ depending continuously on z near 0. What happens to the eigenspaces as $z \to 0$?

Problem 2 (A linear example in 2D)
Let $A \in \mathbb{R}^{2\times 2}$ be a matrix with eigenvalues $\lambda_2 > \lambda_1$ and consider the conservation laws

$$\partial_t u + \partial_x (Au) = 0. \tag{1}$$

1. Explain the concept of simple waves using this example.
2. Show that there is a solution to (1) with initial data

$$u(x,0) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0 \end{cases}$$

   for arbitrary $u_l, u_r \in \mathbb{R}^2$.

3. Illustrate your solution to 2. in the phase plane, i.e. in the (u_1, u_2)-plane where $u = (u_1, u_2)$: u_l, u_r are points in the plane; how are these points connected?

Problem 3 (Rarefaction in shallow water)
Consider the one-dimensional shall-water equations

$$\begin{cases} \phi_t + (v\phi)_x = 0, \\ v_t + \left(\dfrac{v^2}{2} + \phi\right)_x = 0. \end{cases} \tag{2}$$

Remember that this is a system of conservation laws with flux function $\mathbf{F}(z) = (z_1 z_2, \frac{z_2^2}{2} + z_1)$.

1. Show that $\mathbf{r}(z) = (\sqrt{z_1}, 1)$ is a right eigenvector of $D\mathbf{F}$, and solve the ODE $\dot{\mathbf{v}}(s) = \mathbf{r}(\mathbf{v}(s))$ with initial condition $\mathbf{v}(0) = \mathbf{0}$.
2. Compute

$$F(s) = \int_0^s \lambda(\mathbf{v}(t))\, \mathrm{d}t,$$

   for $s > 0$ where $\lambda(z)$ is the eigenvalue of $D\mathbf{F}(z)$ corresponding to $\mathbf{r}(z)$.

3. Let

$$g(x) := \begin{cases} 0, & \text{if } x < 0, \\ 2, & \text{if } x > 0. \end{cases}$$

Find the entropy solution of

$$\begin{cases} w_t + F(w)_x = 0, \\ \qquad w(0,x) = g(x), \end{cases}$$

and use the solution to find a continuous integral solution of (2) with initial conditions

$$v(0,x) = 0, \qquad \phi(0,x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

4. Sketch the graphs of ϕ and v for several values of $t > 0$. What is the physical interpretation of the initial condition and the solution?
