Nonlinear PDEs II — Prof. Dr. J. J. L. Velázquez
Problem Sheet 7 — Dr. M. Helmers, Dr. M. Zaal
Due in class on 19.06.2013

Problem 1 (N-wave)
Let u be the entropy solution to

$$\partial_t u + \frac{1}{p}\partial_x(|u|^p) = 0 \text{ in } \mathbb{R}\times(0,\infty), \qquad u(x,0) = g(x) \text{ on } \mathbb{R},$$

where $p \in (2,\infty)$ and

$$g(x) = \begin{cases} -1, & \text{if } -1 < x < 0, \\ +1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the asymptotic shape $u(\cdot,t)$ as $t \to \infty$.

Problem 2 (Long time asymptotics for periodic initial data)
Let u be the entropy solution to the initial value problem

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \text{ in } \mathbb{R}\times(0,\infty), \qquad u(x,0) = g(x) \text{ on } \mathbb{R},$$

where g is the periodic extension of

$$g(x) = \begin{cases} 0, & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 < x < 1. \end{cases}$$

Show that

$$\|u(\cdot,t) - \overline{g}\|_\infty \to 0 \qquad \text{as } t \to \infty,$$

where $\overline{g} = \frac{1}{2}\int_{-1}^{1} g(x)\,dx = \frac{1}{2}$, and determine the rate of convergence in t. Describe the shape of $u(\cdot,t)$ as $t \to \infty$.

Problem 3 (Strong vs weak solution)
Consider the conservation law

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0$$

1. Show that every strong solution also solves

   $$\partial_t(u^2) + \frac{2}{3}\partial_x(u^3) = 0,$$

   which is a conservation law for u^2 with flux $f(v) = \frac{2}{3}v^{3/2}$.

2. By considering a Riemann problem with $u_l > u_r$ or otherwise, show that the two equations have different weak solutions.

Problem 4 (A linear example)
Consider the system of conservation laws

$$\partial_t u + \partial_x (Au) = 0, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is a diagonalisable matrix with real eigenvalues and $x \in \mathbb{R}$.

1. Find the Rankine-Hugoniot condition along a shock curve. What does it mean?
2. By a linear transformation or otherwise, show that (1) decouples into a system of n independent equations.
