Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 5 Dr. M. Helmers, Dr. M. Zaal
Due in lecture on 27.05.2013

Problem 1 (Conservation)
Let $u \in C_c^0(\mathbb{R} \times [0, \infty])$ be an integral solution to the conservation law

$$\begin{aligned} \partial_t u + \partial_x F(u) &= 0 && \text{in } \mathbb{R} \times (0, \infty], \\ u &= g && \text{on } \mathbb{R} \times \{t = 0\}, \end{aligned}$$

where $g \in C_c^0(\mathbb{R})$ and $F \in C^\infty(\mathbb{R})$, $F(0) = 0$. Show that

$$\int_{\mathbb{R}} u(t, x)\, dx = \int_{\mathbb{R}} g\, dx$$

for all $t > 0$.

Problem 2 (Non-uniqueness in Burger's equation)
Consider the initial-value problem for Burger's equation

$$\begin{cases} \partial_t u + \left(\frac{u^2}{2}\right)^2 = 0 & \text{in } \mathbb{R} \times \{t > 0\}, \\ u(0, x) = g(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

1. It was already shown that

$$u(t, x) = \begin{cases} 1 & \text{if } x \leq t, 0 \leq t \leq 1 \text{ or } x \leq \frac{1+t}{2}, t \geq 1, \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1, 0 \leq t \leq 1, \\ 0 & \text{if } x \geq 1, t \leq 1 \text{ or } x \geq \frac{1+t}{2}, t \geq 1, \end{cases}$$

is a solution in case

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

Define $v(t, x)$ to be $u(t, x)$ for $0 \leq t \leq 2$, and

$$v(t, x) = \begin{cases} 1 & \text{if } x \leq \frac{3}{2} + \sigma_l(t - 2), t \geq 2 \\ v_l & \text{if } \frac{3}{2} + \sigma_l(t - 2) \leq x \leq \frac{1+t}{2}, t \geq 2, \\ v_r & \text{if } \frac{1+t}{2} \leq x \leq t - \frac{1}{2}, t \geq 2, \\ 0 & \text{if } x \geq t - \frac{1}{2}, t \geq 2. \end{cases}$$

Use the Rankine-Hugoniot condition to find $\sigma_l < \frac{1}{2}$, $v_l \neq 1$ and $v_r \neq 0$ such that v is an integral solution of (1). Sketch the curves of discontinuities of v together the projected characteristics.

2. Use a similar construction to find a nonzero integral solution of (1) for $g \equiv 0$. (Hint: construct a solution with three curves of discontinuities starting at $x = 0$)

Problem 3 (Travelling waves as solutions of regularised Burgers' equation)
Let $\sigma_0 \in \mathbb{R}$ and let

$$U_0(z) = \frac{1}{1+e^{z/2}}.$$

1. Under what conditions on a curve $\sigma(t)$, $\sigma(0) = \sigma_0$ is

   $$u_\varepsilon(t,x) = U_0\left(\frac{x-\sigma(t)}{\varepsilon}\right), \qquad 0 < \varepsilon < 1$$

   a solution to the regularised Burgers' equation

   $$\partial_t u + \frac{1}{2}\partial_x(u^2) = \varepsilon \partial^2_{xx} u?$$

2. Find

   $$u(t,x) = \lim_{\varepsilon \to 0} u_\varepsilon(t,x)$$

   and the asymptotic problem that is solved by u. In what sense does u solve the problem?
