Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 4 Dr. M. Helmers, Dr. M. Zaal
Due in class on 15.05.2013

Problem 1 (Hopf's construction for Hamilton–Jacobi equations)
Recall from the lecture that for any $y, q \in \mathbb{R}^n$ the affine function

$$v_{y,q}(t,x) = g(y) + q \cdot (x-y) - tH(q)$$

solves the Hamilton–Jacobi equation in the initial value problem

$$\begin{aligned} \partial_t u + H(Du) &= 0 && \text{in } \mathbb{R}^n \times \{t>0\}, \\ u(0,x) &= g(x) && \text{on } \mathbb{R}^n \times \{t=0\}, \end{aligned} \tag{1}$$

but does in general not satify the initial condition. Let H be smooth, convex and coercive, and let g be Lipschitz. Show that the envelope

$$v(t,x) = \inf_{y\in\mathbb{R}^n} \sup_{q\in\mathbb{R}^n} v_{y,q}(t,x)$$

solves the initial value problem (1).

Problem 2 (Nonconvex Hamiltonian)
Let $W(p) = (1-p^2)^2$ be a double-well and consider the initial value problem

$$\begin{aligned} \partial_t u + W(\partial_x u) &= 0 && \text{in } \mathbb{R} \times \{t>0\}, \\ u(0,x) &= g(x) && \text{on } \mathbb{R} \times \{t=0\}, \end{aligned} \tag{2}$$

with Lipschitz continuous initial data $g\colon \mathbb{R} \to \mathbb{R}$. If W was convex, the Hopf–Lax formula

$$u(t,x) = \inf_{y\in\mathbb{R}} \left\{ tW^*\left(\tfrac{x-y}{t}\right) + g(y) \right\}, \tag{3}$$

where W^* is the Legendre transform of W, would give a solution to (2). Now, since W is not convex, what problem does (3) solve according to the theory of the lecture, that is, what is the Hamilton–Jacobi equation that corresponds to the minimisation problem for W^*?
Hint: Recall the proof of convex duality. You do not need to compute W^* as intermediate result.

Problem 3 (Discontinuous initial data and Hopf-Lax formula)
Consider the initial value problem

$$\begin{aligned} \partial_t u + \frac{1}{2}|\partial_x u|^2 &= 0 && \text{in } \mathbb{R} \times \{t>0\}, \\ u(0,x) &= \chi_{\{y<0\}}(x) && \text{on } \mathbb{R} \times \{t=0\}, \end{aligned} \tag{4}$$

where

$$\chi_{\{y<0\}}(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0 \end{cases}$$

is the characteristic function of the set $\{y<0\}$.

1. Let $u = u(t, x)$ be given by the Hopf–Lax formula. Show that

$$u(t, x) = \phi(x/\sqrt{t})$$

for some function ϕ. Find ϕ and u. Does u solve (4) in some sense?

2. Using the ansatz $u(t, x) = \psi(x/\sqrt{t})$ for some function $\psi(y)$ in (4), find a boundary value problem for ψ and validate that your ϕ from 1. is a solution in some sense.
