Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 3 Dr. M. Helmers, Dr. M. Zaal
Due in class on 08.05.2013

Problem 1 (Legendre transform)
Denote by $L = H^*$ the Legendre transform of a convex function $H\colon \mathbb{R}^n \to \mathbb{R}$.

1. Let $H(p) = \frac{1}{r}|p|^r$ for some $1 < r < \infty$. Show that
$$L(q) = \frac{1}{r'}|q|^{r'} \qquad \text{where} \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

2. Let
$$H(p) = \frac{1}{2}p^T Ap + b^T p = \frac{1}{2}\sum_{i,j=1}^{n} p_i a_{ij} p_j + \sum_{i=1}^{n} b_i p_i$$
where $A = (a_{ij})$ is a symmetric, positive definite matrix and $b \in \mathbb{R}^n$. Compute L.

Hint: In 1. you might want to use Young's inequality $ab \le \frac{1}{r}a^r + \frac{1}{r'}b^{r'}$ for $a, b \ge 0$ and r, r' as above.

Problem 2 (L^∞-contraction property of Hamilton-Jacobi equations)
Let u^1, u^2 be solutions to

$$\begin{aligned} u^i_t + H(Du^i) &= 0 && \text{in } \mathbb{R}^n \times (0,\infty), \\ u^i &= g^i && \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

that are given by the Hopf-Lax formula ($i = 1, 2$). Assume that $H \in C^2(\mathbb{R}^n)$ is uniformly convex and has superlinear growth, and let g^1, g^2 be Lipschitz. Show that
$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty} \le \|g^1 - g^2\|_{L^\infty}$$
for all $t > 0$.

Problem 3 (Hopf-Lax formula I)
Let $E \subset \mathbb{R}^n$ be closed. Show that if the Hopf-Lax formula could be applied to the initial value problem

$$\begin{aligned} u_t + |Du|^2 &= 0 && \text{in } \mathbb{R}^n \times (0,\infty), \\ u &= \begin{cases} 0 & \text{if } x \in E \\ \infty & \text{if } x \notin E \end{cases} && \text{on } \mathbb{R}^n \times \{t = 0\}, \end{aligned} \tag{1}$$

it would give the solution

$$u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2. \tag{2}$$

Is (2) a reasonable solution to (1)? Note that, if ∂E is sufficiently smooth, there is a neighbourhood U of E such that $\operatorname{dist}(\cdot,E)$ is differentiable almost everywhere in U with $|D\operatorname{dist}(x,E)| = 1$.

Problem 4 (Hopf-Lax formula II)
Consider the problem

$$\begin{aligned} u_t + |u_x| &= 0 \quad \text{in } \mathbb{R} \times \{t > 0\}, \\ u &= g \quad \text{on } \mathbb{R} \times \{t = 0\}, \end{aligned} \tag{3}$$

where

$$g(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 1, \\ 1 + x & \text{if } -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using formal considerations in the Hopf-Lax formula, find a solution to (3) that satisfies the equation almost everywhere and attains the initial data. Is your solution a weak solution in the sense defined in the lecture?
