Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 2 Dr. M. Helmers, Dr. M. Zaal
Due in lecture on 02.05.2013

Problem 1 (A quasilinear PDE)
Consider the equation

$$x^2\partial_x u + (y+1)\partial_y u = 0. \tag{1}$$

1. Find a solution to (1) in some neighbourhood of the x-axis in $\mathbb{R}^2$ that satisfies $u(x,0) = x$.

2. Prove that if $u \in C^1(\mathbb{R}^2)$ is a solution to (1) for all $(x,y) \in \mathbb{R}^2$, then $u(x,0)$ is constant for $x \geq 0$.

Problem 2 (Another nonlinear PDE)
Consider the equation

$$u_y = u_x^3 \qquad \text{in } \mathbb{R}^2. \tag{2}$$

1. Find a solution with $u(x,0) = 2x^{3/2}$ near $\Gamma = \{(x,0) : x \in \mathbb{R}\}$.

2. Show that every solution to (2) that is regular for all $(x,y) \in \mathbb{R}^2$ is linear.

Problem 3 (Picone)
Let $u \in C^1(\overline{\Omega})$ be a solution to

$$a(x,y)\partial_x u + b(x,y)\partial_y u = -u,$$

where $\Omega = B_1(0)$ is the open unit ball in $\mathbb{R}^2$ and $a, b \in C^\infty(\mathbb{R}^2)$. Assume that $a(x,y)x + b(x,y)y > 0$ on $\partial\Omega$ and prove $u \equiv 0$ in Ω.

Problem 4 (Blow up in Burgers' equation)
Let $u_0 \in C^\infty(\mathbb{R})$ be such that u_0' is bounded and has a unique minimiser s_0 with $u_0'(s_0) < 0$, $u_0''(s_0) = 0$, and $u_0'''(s_0) > 0$. Set

$$u(t,x) = u_0(\xi), \qquad x = u_0(\xi)t + \xi.$$

1. Show that there is $0 < t^* < \infty$ such that u solves the equation

$$\partial_t u + u\partial_x u = 0 \qquad \text{for } t \in (0,t^*),\ x \in \mathbb{R} \tag{3}$$

   and satisfies the initial condition $u(0,x) = u_0(x)$. Determine the largest t^*, which depends on some $\xi^* \in \mathbb{R}$, such that u as a function of (t,x) exists and solves (3). What happens with $\partial_t u$, $\partial_x u$ and the characteristic curves of (3) as $t \to t^*$?

2. Perform a local analysis (Taylor expansion, term comparison) to argue that

$$u(t,x) - u_0(\xi^*) \sim -(x - x^*)^{1/3}$$

   close to (t^*, x^*), where $x^* = u_0(\xi^*)t^* + \xi^*$. Here $a \sim b$ means $a = Cb$ up to some higher order terms.
