Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 1 Dr. M. Helmers, Dr. M. Zaal
Due in class on 24.04.2013

Problem 1 (1D Clairot equation)
Recall from the lecture that the lines $u = ax + b$ are affine solutions to the one-dimensional Clairot equation $u = xu' + g(u')$ provided that $g(a) = b$.

1. Suppose that $u \in C^2(I)$ is an arbitrary function on an interval $I \subset \mathbb{R}$ such that $u'' > 0$ (or $u'' < 0$) in I. Show that the mapping $\xi = u'(x)$ is invertible, and that
$$u(x) + \eta(\xi) = x \cdot \xi, \qquad \xi = u'(x), \qquad x = \eta'(\xi)$$
for some function η. (This is called *Legendre transform.*)

2. Use 1. to obtain a solution of Clairot's equation in parametric form, i. e. in the form $x = x(t)$, $u = u(t)$ for some parameter t, provided that $g'' \neq 0$.

3. Find a nonlinear solution to
$$u = xu' - \frac{cu'}{\sqrt{1 + u'^2}}$$
in parametric and non-parametric form, where $c > 0$ is a constant.

Problem 2 (Linear transport in 1D)
Consider the linear transport equation
$$\partial_t u(t, x) + a(t, x)\partial_x u(t, x) = 0. \tag{1}$$

1. Determine the characteristic curves $x = x(t)$, $x(0) = x_0$ for the transport velocities
(a) $a(t, x) = x$,
(b) $a(t, x) = \omega \cosh(\omega t + \phi)$, $\omega, \phi \in \mathbb{R}$,
(c) $a(t, x) = g(x_0)$, where
$$g(x_0) = \begin{cases} 0 & \text{if } x_0 \leq 0, \\ -x_0 & \text{if } 0 < x_0 < 1, \\ -1 & \text{if } 1 \leq x_0. \end{cases}$$
In each case, sketch the trajectories $x(t)$ for $t \geq 0$ and several values of x_0. What happens in (c) for t sufficiently large, and how can this behaviour be interpreted physically, e. g. if $x(t)$ are trajectories of gas molecules?

2. Give a simple yet non-trivial condition on $a = a(t, x)$ such that
(a) characteristic curves do not intersect,
(b) a solution to (1) exists for all $t > 0$.

Problem 3 (A nonlinear PDE)
Use the method of characteristics to solve
$$\begin{aligned} u_{x_1} u_{x_2} &= x_1 x_2 && \text{in } \mathbb{R}^2, \\ u(x_1, x_2) &= x_1 && \text{on } \Gamma = \{x_2 = 0\}. \end{aligned}$$

Problem 4 (Scalings of Burgers' equation)
Obviously, Burgers' equation

$$\partial_t u + u\,\partial_x u = 0. \tag{2}$$

is invariant under the transformation $x \mapsto x - x_0$, $t \mapsto t - t_0$, that is, if u solves (2), then also $\hat{u}(t,x) = u(t - t_0, x - x_0)$ solves (2).

1. Given $u_0 \in \mathbb{R}$, show that the transformation $x \mapsto x - u_0 t$, $u \mapsto u + u_0$ leaves (2) invariant.

2. Given a solution u of (2), show that $u_\lambda(t,x) = \lambda^\alpha u(\lambda^\beta t, \lambda^\gamma x)$ solves (2) for any $\lambda > 0$, provided $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy certain relations.
