Problem 2 (Long time asymptotics for periodic initial data)
Let u be the entropy solution to the initial value problem

$$\partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \text{ in } \mathbb{R} \times (0, \infty), \qquad u(x, 0) = g(x) \text{ on } \mathbb{R},$$

where g is the periodic extension of

$$g(x) = \begin{cases} 0, & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 < x < 1. \end{cases}$$

Show that

$$\|u(\cdot, t) - \overline{g}\|_\infty \to 0 \qquad \text{as } t \to \infty,$$

where $\overline{g} = \frac{1}{2}\int_{-1}^{1} g(x)\,dx = \frac{1}{2}$, and determine the rate of convergence in t. Describe the shape of $u(\cdot, t)$ as $t \to \infty$.

This can be solved by proving the following extension of Theorem 5 from Evans, §3.4. In the proof of this theorem, it is shown that

$$\left|\frac{x - y(x,t)}{t} - \sigma\right| \le \frac{C}{\sqrt{t}},$$

where $y(x, t)$ is defined by the Lax-Oleinik formula. Intuitively, this estimate means that the characteristics of the equation will move more or less at speed σ for large t.
In the case studied by Evans, $\sigma = F'(0)$. The following theorem follows from a similar argument: it is shown that, asymptotically, characteristics move at a certain speed, which is in general not equal to $F'(0)$.

Theorem 1 (Asymptotics for periodic initial conditions)
Suppose that F is smooth, uniformly convex, $F(0) = 0$ and g is bounded and periodic. Then

$$|u(x,t) - \overline{g}| \le \frac{C}{\sqrt{t}} \tag{1}$$

Proof. Let $\theta := \inf_{x\in\mathbb{R}} F''(x)$. Since F is uniformly convex, $\theta > 0$. Similar to the proof in Evans,

$$tL\left(\frac{x-y}{t}\right) \ge tL(\sigma) + L'(\sigma)\,(x - y - \sigma t) + \theta\frac{|x - y - \sigma t|^2}{t},$$

where σ is to be determined. Moreover, for $h(y) = \int_0^y g(x)\,\mathrm{d}x$ we have

$$\overline{g}y - M \le h(y) \le \overline{g}y + M$$

where M is the integral of $|g - \overline{g}|$ over one period. It follows that

$$tL\left(\frac{x-y}{t}\right) + h(y) \ge tL(\sigma) + L'(\sigma)\,(x - y - \sigma t) + \theta\frac{|x - y - \sigma t|^2}{t} + \overline{g}y - M.$$

On the other hand,

$$tL\left(\frac{x-(x-\sigma t)}{t}\right) + h(x-\sigma t) \le tL(\sigma) + \overline{g}(x-\sigma t) + M.$$

For $y(x,t)$ we now have

$$\begin{aligned} tL(\sigma) + L'(\sigma)\,(x - y(x,t) - \sigma t) + \theta\frac{|x-y(x,t)-\sigma t|^2}{t} + \overline{g}y(x,t) - M \\ \le tL(\sigma) + \overline{g}(x-\sigma t) + M, \end{aligned}$$

that is,

$$(L'(\sigma) - \overline{g})\,(x - y(x,t) - \sigma t) + \theta\frac{|x-y(x,t)-\sigma t|^2}{t} \le 2M.$$

Setting $\sigma := (L')^{-1}(\overline{g}) = F'(\overline{g})$ now implies

$$\left|\frac{x-y(x,t)}{t} - F'(\overline{g})\right| \le \frac{2M}{\theta\sqrt{t}}.$$

Proceeding as in Evans' proof,

$$\begin{aligned} |u(x,t)-\overline{g}| &= \left|G\left(\frac{x-y(x,t)}{t}\right) - G(F'(\bar{}))\right| \\ &\le \mathrm{Lip}(G)\left|\frac{x-y(x,t)}{t} - F'(\overline{g})\right|, \end{aligned}$$

which implies the result. $\square$

In the example of the problem, F' and G are the identity, $\theta = 1$, $\overline{g} = \frac{1}{2}$ and $M = 1$ (in fact, the estimates also hold for $M = \frac{1}{2}$). Therefore,

$$|u(x,t) - \tfrac{1}{2}| \le \frac{1}{\sqrt{t}} \tag{2}$$
