Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Construction of a solution to a Riemann problem from elementary waves
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Problem 1 (Waves in shallow water – 8 MARKS)
Consider the one-dimensional shall-water equations

$$\begin{cases} \phi_t + (v\phi)_x = 0, \\ v_t + \left(\dfrac{v^2}{2} + \phi\right)_x = 0, \end{cases} \tag{1}$$

which is a system of conservation laws with flux function $\mathbf{F}(z) = (z_1 z_2, \frac{z_2^2}{2} + z_1)$. In the first four parts of this problem, it was shown that

- The eigenvalue/eigenvector pairs of $D\mathbf{F}$ are given by

$$\lambda_{1,2}(\phi, v) = v \pm \sqrt{\phi}, \qquad \mathbf{r}_{1,2} = \begin{pmatrix} \pm\sqrt{\phi} \\ 1 \end{pmatrix}, \tag{2}$$

  where the index 1 corresponds to the '$-$' solution.

- The rarefaction curves are given by

$$R_{1,2}(\phi_0, v_0) = \left\{ \mathbf{v}_{1,2}(s) := \begin{pmatrix} (\sqrt{\phi_0} \pm \frac{s}{2})^2 \\ v_0 + s \end{pmatrix} \right\}, \tag{3}$$

  where again the index 1 corresponds to the '$-$'. In both cases, the positive direction is the part parametrized by $s > 0$.

- The shock curves going through $(\eta, 0)$ are given by

$$S_{1,2}(\eta, 0) = \left\{ (\phi, v) \in \mathcal{S} : \phi = \eta + \frac{v^2}{2} \pm v\sqrt{\eta + \frac{v^2}{16}} \right\}. \tag{4}$$

  As before, the index 1 corresponds to the '$-$' solution, and the positive direction is parametrized by $v > 0$.

- If $\xi = 7 + \sqrt{6} = (1 + \sqrt{6})^2$,

$$R_1^+(\xi, 0) \cap S_2^+(2, 0) \ni \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \text{ with } \sigma = 3 \tag{5}$$

5. Use the results to construct an integral solution of (1) with initial conditions

$$v(0, x) = 0, \qquad \phi(0, x) = \begin{cases} \xi, & \text{if } x < 0, \\ 2, & \text{if } x > 0. \end{cases}$$

   Set $u_l := (\xi, 0)$, $u_c = (6, 2)$ and $u_r = (2, 0)$. The first step is to construct an integral solution of the shallow water equations with initial condition u_l for

$x < 0$ and u_c for $x > 0$. Since $u_c \in R_1(u_l)$, this solution will be a rarefaction wave:

$$\mathbf{u}_1(x,t) = \mathbf{v}_1(w(x,t)) \tag{6}$$

where w is an entropy solution of

$$w_t + f(w)_x = 0, \text{ with } f(s) = \int_0^s \lambda(\mathbf{v}_1(t))\,\mathrm{d}t = \frac{3s^2}{4} - \sqrt{\xi}s \tag{7}$$

and initial condition 0 for $x < 0$, 2 for $x > 0$. Since $\lambda_1(u_l) = -\sqrt{\xi}$ and $\lambda_1(u_c) = 2 - \sqrt{6}$,

$$w(x,t) = \begin{cases} 0, \text{if } \frac{x}{t} < -\sqrt{\xi}, \\ \frac{2}{3}(\frac{x}{t} + \sqrt{\xi}), & \text{if } -\sqrt{\xi} < \frac{x}{t} < 2 - \sqrt{6}, \\ 2, & \text{if } \frac{x}{t} > 2 - \sqrt{6}, \end{cases} \tag{8}$$

that is

$$\mathbf{u}_1(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} < -\sqrt{\xi}, \\ \mathbf{v}_1\left(\frac{2}{3}(\frac{x}{t} + \sqrt{\xi})\right), & \text{if } -\sqrt{\xi} < \frac{x}{t} < 2 - \sqrt{6}, \\ u_c, & \text{if } \frac{x}{t} > 2 - \sqrt{6}. \end{cases} \tag{9}$$

Since, by construction $u_r \in S_2^-(u_c)$, an integral solution $\mathbf{u}_2$ of the shallow water equations with initial condition u_c for $x < 0$ and u_r for $x > 0$) can be constructed:

$$\mathbf{u}_2(x,t) = \begin{cases} u_c, & \text{if } \frac{x}{t} < 3, \\ u_r, & \text{if } \frac{x}{t} > 3. \end{cases} \tag{10}$$

Note that, because $\lambda_1(\xi, 0) < \lambda_1(6,2) < \sigma$, the solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ are the same whenever $\frac{x}{t}$ is between $\lambda_1(6,2) = 2 - \sqrt{6}$ and $\sigma = 3$. Therefore, we know that

$$\mathbf{u}(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} < -\sqrt{\xi}, \\ \mathbf{v}_1\left(\frac{2}{3}(\frac{x}{t} + \sqrt{\xi})\right), & \text{if } -\sqrt{\xi} < \frac{x}{t} < 2 - \sqrt{6}, \\ u_c, & \text{if } 2 - \sqrt{6} < \frac{x}{t} < 3, \\ u_r, & \text{if } \frac{x}{t} > 3. \end{cases} \tag{11}$$

must be an integral solution of the shallow water equations. By construction the initial condition is u_l for $x < 0$ and u_r for $x > 0$.
