Sheet 1

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1 Introduction

The aim of this course is to introduce a series of ideas related to the robust analysis of solutions to differential equations driven by irregular signals and related problems. Initiated by work of T. Lyons in the '90 these ideas goes under the name of rough path theory. A rough path is an object which lives on top of a given continuous path on a Banach space (which can be infinite dimensional). This object, in some sense, encodes informations about the “microscopic” behaviour of the path, information which in general is not possible to recover in a robust (continuous) way from the knowledge of the path alone. This lack of robustness is essentially due to the low regularity of the path. In some sense we will trade regularity of the path itself with some analytic information on this additional structure. On a general perspective our task can be considered that of extending natural non–linear operations to spaces of distributions (in the sense of L. Schwartz) in a continuous way. This is of course not possible using only analytical tools in standard spaces of distributions since there are easy counterexample at hand. The key new idea is to identify particular sets of distributions (or functions) on which we possess enough information to make our operations continuous. In order to keep the exposition at a reasonable level, away from technical difficulties, we will restrict ourselves to analyse the case where the irregular behaviour depends on a single independent real variable which will be mostly the time variable. The prototypical problem of the difficulties which can arise while mixing non–linear operations and distributions is the analysis of the continuity of the intergral map given by

$$\begin{array}{rcl} (f,g) & \mapsto & I(f,g) := \int_0^{\cdot} f(s)\partial_s g(s)\mathrm{d}s \\ C^\alpha \times C^\beta & \to & C^\gamma \end{array}$$

where C^ζ denotes the space of ζ–Hölder continuous real functions in [0, 1]. This map is well defined for f continuous and g continuously differentiable and we ask if it is possible to extend it in a continuous way to all $(f,g) \in C^\alpha \times C^\beta$ for certain values of the Hölder exponents α, β.

That this extension is not possible in general is clear from the following counterexample. Let

$$f_n(t) = n^{-\alpha}\cos(2\pi n t), \quad g_n(t) = n^{-\alpha}\sin(2\pi n t),$$

for some fixed $\alpha \in (0,1)$ and $n \geqslant 1$. Then if we consider the Hölder semi–norm

$$\|h\|_\zeta = \sup_{0 \leqslant s \leqslant t \leqslant 1} \frac{|h(t) - h(s)|}{|t-s|^\zeta}$$

we have

$$\|f_n\|_\zeta + \|g_n\|_\zeta \to 0$$

as $n \to \infty$ whenever $0 < \zeta < \alpha$. However

$$I(f_n, g_n)(t) = -2\pi n^{1-2\alpha} \int_0^t \cos^2(2\pi n s) \mathrm{d}s = -2\pi n^{1-2\alpha} \left(\frac{t}{2} + o(1)\right)$$

uniformly in $t \in [0, 1]$. Which means that $I(f_n, g_n)(1) \to I(0, 0)(1) = 0$ only when $2\alpha > 1$. This example shows that the integral map $(f, g) \mapsto I(f, g)(1)$ cannot be continuous from $C^\alpha \times C^\alpha$ if $\alpha < 1/2$. In general is easy to see that the map cannot be continuous from $C^\alpha \times C^\beta$ if $\alpha + \beta < 1$.

The fact that, on the other hand, the integral map is continuous from $C^\alpha \times C^\beta \to C^\beta$ when $\alpha + \beta > 1$ is already a quite useful result, as we will show during these lectures. The map I_Y which one obtains by such an extension is referred to as the Young integral. Originally Young proved this result in a topology different from the Hölder one but for the moment we will not discuss further this issue. The Hölder topology is quite convenient for many of our considerations and we will stick to it for most of our exposition. Construction of the Young integral and some of its applications will be our first goal. Rough paths will appear only later, when we will try to go beyond Young's theory.

2 The sewing map

There are various ways to construct the Young integral, here we will follow a longish path which however will build up the basic tool to handle also the rough path case. This tool is the sewing map and the related sewing lemma.

In order to introduce this map we would like to have an algebraic characterization of the (indefinite) integral $I(f, g)$ of the integrand f with respect to the integrator g, characterization which do not use differential calculus (since we hope to extend our construction to non–differentiable objects). From a differentiable point of view, the integral $I(f, g)$ is just the unique solution of the differential equation $\partial_t I(f, g)(t) = f(t)\partial_t g(t)$. Going from derivatives to finite increments we could say that the integral $I(f, g)$ is that particular function I which satisfy

$$I(t) - I(s) = f(s)(g(t) - g(s)) + o_u(|t - s|), \qquad I(0) = 0, \tag{1}$$

for any $0 \leqslant s \leqslant t \leqslant 1$, where the remainder is $o_u(|t - s|)$ uniformly in s, t. It is clear that this property is satisfied by the usual integral as soon as g is at least C^1. Moreover this equation provides a characterization of the integral since there cannot be another function J (necessarily continuous) with the same property, since in this case the difference $D = I - J$ would satisfy

$$D(t) - D(s) = o_u(|t - s|)$$

but this means that $D(t) = D(s)$ for all $0 \leqslant s \leqslant t \leqslant 1$ and so $D(t) = 0$ for all $t \in [0, 1]$. Eq. (1) provides then an alternative description of the integral map. We can look at it as saying that the integral I is the only function whose increments match the "germ" $f(s)(g(t) - g(s))$ modulo a negligible error. Underlying this picture there is a cochain complex of n-increments defined as follows: an n-cochain ($n \geqslant 1$) is an element of $C_n(V) = C(\Delta_n; V)$ where $\Delta_n = \Delta_n(0, 1)$ and

$$\Delta_n(s, t) = \{(s_1, \ldots, s_n): s \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant t\}$$

where V is a vector space (for the moment just consider $V=\mathbb{R}$) and the coboundary $\delta: C_n(V) \to C_{n+1}(V)$ is given by

$$\delta f(s_1, \ldots, s_{n+1}) = \sum_{k=1}^{n+1} (-1)^{n+1-k+1} f(s_1, \ldots, \cancel{s_k}, \ldots, s_{n+1})$$

so for example

$$\delta f(s,t) = f(t) - f(s), \qquad \delta f(s,u,t) = f(s,t) - f(u,t) - f(s,u).$$

The basic property of the coboundaries is $\delta \circ \delta = 0$. This defines the complex

$$\mathbb{R} \to C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_3 \xrightarrow{\delta} C_4 \xrightarrow{\delta} \cdots$$

where the first arrow is given by the constant map. This complex is exact: if $\delta f = 0$ then there exists g such that $f = \delta g$ (find it). Letting $A(s,t) = f(s)\delta g(s,t) = f(s)(g(t) - g(s))$ we have $A = \delta I + R$ where $R(s,t) = o_u(|t-s|)$. So we see that R (and thus I) is obtained by computing a particular representative of the class of A in the cohomology of the above complex, namely the representative such that $R(s,t) = o_u(|t-s|)$. Given two arbitrary functions f, g it is not clear that this is always possible. But whenever it is possible we can try to define the integral of f with respect to g (modulo constants) from the formula $A = \delta I + R$. Now the class of A can be identified with $\delta A \in C_3$. The *sewing map* provides, under certains smallness conditions, the way to recover R from δA.

In order to formulate a precise statement we need to introduce topologies on C_n. We say that $h \in C_n^\alpha$ if

$$\|h\|_{\alpha,[s,t]} := \sup_{(s_1,\ldots,s_n) \in \Delta_n(s,t)} \frac{|h(s_1, \ldots, s_n)|}{|s_n - s_1|^\alpha} < +\infty.$$

And we let $C_n^{\alpha+} = \cup_{\beta > \alpha} C_n^\beta$. Remark that $\delta C_1 \cap C_2^{1+} = \{0\}$.

Theorem 1. *(Sewing map) There exists a unique map $\Lambda: C_3^{1+} \cap \delta C_2 \to C_2^{1+}$ such that $\delta\Lambda = \mathrm{Id}_{C_3}$. It satisfies*

$$\|\Lambda \delta A\|_{z,I} \leqslant \frac{2^z}{2^z - 2} \|\delta A\|_{z,I}$$

for all $z > 1$ and closed interval $I \subseteq \mathbb{R}$.

Proof. Assume we already showed the existence of a map Λ such that $\delta\Lambda = \mathrm{Id}_{C_3}$ and for which

$$|(\Lambda \delta A)(s,t)| \leqslant C\,|t-s|^\rho, \qquad (s,t) \in \Delta_2$$

for some $\rho > 1$ (actually less is necessary) then we show that also the claimed estimate for Λ holds. We just note that for all $s, t \in I$ we have

$$|(\Lambda\delta A)(s,t)| \leqslant \sum_{i=0}^{2^n-1} |(\Lambda\delta A)(t_i^n, t_{i+1}^n)| + \sum_{k=0}^{n-1} \sum_{i=0}^{2^k-1} |\delta(\Lambda\delta A)(t_{2i}^{k+1}, t_{2i+1}^{k+1}, t_{2i+2}^{k+1})|$$

where $\{t_i^n = s + 2^{-n}(t-s)\,i, i = 0, \ldots, 2^n\}$ is a dyadic partition of $[s,t]$. Direct estimations give

$$|(\Lambda\delta A)(s,t)| \leqslant C 2^{n(1-\rho)} + \|\delta A\|_{z,I} \sum_{k=0}^{n-1} 2^k 2^{-kz} |t-s|^z \leqslant C 2^{n(1-\rho)} + \|\delta A\|_{z,I} \frac{1}{1-2^{1-z}} |t-s|^z$$

and sending $n \to \infty$ we obtain

$$|(\Lambda\delta A)(s,t)| \leqslant \frac{1}{1-2^{1-z}} \|\delta A\|_{z,I} |t-s|^z, \qquad s,t \in I.$$

Let us prove now the existence of such a map Λ. Fix a smooth function $Q\colon \mathbb{R} \to \mathbb{R}_+$ with compact support in $\mathbb{R}_+$ and unit integral and let $Q_\sigma(x) = Q(\sigma^{-1}x)\sigma^{-1}$. Extend A to $\mathbb{R} \times \mathbb{R}$ so that $A(s, t) = A(J_{0,1}(s), J_{0,1}(t))$ where $J_{a,b}(r) = \min(a, \max(r, b))$. For $\sigma > 0$ and $0 \leqslant s \leqslant t \leqslant 1$ define a approximation A_σ to A which is smooth in the second variable:

$$A_\sigma(s,t) = \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma'(r'') A(s, r+r'') = \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma(r'')(A(s,t+r'') - A(s,s+r''))$$

where $Q_\sigma'(x) = \partial_x Q_\sigma(x)$. This strange definition is motivated by the fact that we want $A_\sigma(s,s) = 0$. Note that

$$\delta A_\sigma(s,u,t) = \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma(r'')(\delta A(s,u,t+r'') - \delta A(s,u,u+r''))$$

and that $A_\sigma \to A$ and $\delta A_\sigma \to \delta A$ pointwise. Now let

$$(\mathscr{R} A_\sigma)(s,t) = \int_s^t \partial_2 A_\sigma(r,r)\mathrm{d}r = \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma'(r'') A(r, r+r'')$$

where $\partial_2 A_\sigma(s,t) = \partial_t A_\sigma(s,t)$ and set $\Lambda\delta A_\sigma = A_\sigma - \mathscr{R} A_\sigma$. Then

$$(\Lambda\delta A_\sigma)(s,t) = \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma'(r+r'')(A(s,r'') - A(r,r'')) = \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}r''\, Q_\sigma'(r'') \delta A(s,r,r+r'')$$

and $\delta(\Lambda\delta A_\sigma)(s,u,t) = \delta A_\sigma(s,u,t)$. When $|t-s| \leqslant \sigma$ we can estimate

$$|(\Lambda\delta A_\sigma)(s,t)| \lesssim |t-s|\sigma^{-1}(|t-s|+\sigma)^z \lesssim |t-s|\sigma^{z-1}.$$

While when $|t-s| > \sigma$ by dyadically refining at each step the interval $[s,t]$ we have

$$(\Lambda\delta A_\sigma)(s,t) = \sum_{i=0}^{2^n-1} (\Lambda\delta A_\sigma)(t_i^n, t_{i+1}^n) + \sum_{k=0}^{n-1} \sum_{i=0}^{2^k-1} \delta A_\sigma(t_{2i}^{k+1}, t_{2i+1}^{k+1}, t_{2i+2}^{k+1})$$

Choosing n such that $|t-s|2^{-n} \leqslant \sigma < |t-s|2^{-n+1}$ together with the two estimates

$$|(\Lambda\delta A_\sigma)(t_i^n, t_{i+1}^n)| \lesssim |t-s|2^{-n}\sigma^{z-1} \lesssim |t-s|^z 2^{-nz}$$

and for $k<n$

$$|\delta A_\sigma(t_{2i}^{k+1},t_{2i+1}^{k+1},t_{2i+2}^{k+1})|\lesssim(|t-s|2^{-k}+\sigma)^z\lesssim 2^{-zk}|t-s|^z(1+2^{k-n})^z\lesssim 2^{-zk}|t-s|^z$$

gives

$$|(\Lambda\delta A_\sigma)(s,t)|\lesssim|t-s|^z\left[2^{n(1-z)}+\sum_{k=0}^{n-1}2^{k(1-z)}\right]\lesssim|t-s|^z\lesssim|t-s|$$

so

$$\sup_{\sigma>0}\sup_{0\leqslant s<t\leqslant 1}\frac{|(\Lambda\delta A_\sigma)(s,t)|}{|t-s|}\lesssim 1.$$

Now when $|t-s|>\sigma$

$$|A_\sigma(s,t)|\leqslant|t-s|^\gamma+(|t-s|+\sigma)^z\leqslant|t-s|^\gamma+|t-s|^z\lesssim|t-s|^\gamma$$

and when $|t-s|\leqslant\sigma$

$$|A_\sigma(s,t)|\lesssim\sigma^{-1}|t-s|(|t-s|+\sigma)^\gamma\lesssim|t-s|\sigma^{\gamma-1}\lesssim|t-s|^\gamma.$$

So

$$\sup_{\sigma>0}\sup_{0\leqslant s<t\leqslant 1}\frac{|(\mathscr{R}A_\sigma)(s,t)|}{|t-s|^\gamma}\lesssim 1$$

which means that the sequence of functions $f_\sigma(t)=\mathscr{R}A_\sigma(0,t)$ is uniformly continuous. By choosing a converging subsequence denoted f_n we obtain a limit f which we call $(\mathscr{R}A)(0,t)=f(t)$. Then noting that $A_\sigma\to A$ pointwise and letting $\Lambda\delta A=\mathscr{R}A-A$ we have $(\Lambda\delta A)(s,t)=\lim_{\sigma\to 0}(\Lambda\delta A_\sigma)(s,t)$ and the key estimate

$$|(\Lambda\delta A)(s,t)|=\lim_{n\to\infty}|(\mathscr{R}A_{\sigma_n}-A_{\sigma_n})(s,t)|\lesssim|t-s|^z$$

which almost concludes the proof of existence since $\delta(\Lambda\delta A_\sigma)=\delta A_\sigma\to\delta A$ pointwise by the continuity of A. □

Alternative proof of the sewing lemma. Other proofs of existence of the sewing map proceed via discrete approximations. For example, letting $\{u_i^n\}$ be the n-th order dyadic partition of $[0,1]$ one can let

$$\mathcal{S}_nA(t)=\sum_{i=0}^{2^n-1}\mathbb{I}_{t\geqslant u_i}A(u_i^n,u_{i+1}^n\wedge t)$$

and estimating

$$|\delta\mathcal{S}_nA(s,t)-A(s,t)|\leqslant\sum_{k=\ell+1}^{n}|\delta\mathcal{S}_kA(s,t)-\delta\mathcal{S}_{k-1}A(s,t)|$$

where ℓ is the greatest integer such that $2^{-\ell}\geqslant|t-s|$. The differences can be readily controlled by

$$|\delta\mathcal{S}_kA(s,t)-\delta\mathcal{S}_{k-1}A(s,t)|\lesssim 2^{k-\ell}2^{-zk}\|\delta A\|_z$$

and one obtains

$$|\delta\mathcal{S}_n A(s,t) - A(s,t)| \lesssim \|\delta A\|_z 2^{-\ell} \sum_{k=\ell+1}^{\infty} 2^{k(1-z)} \lesssim \|\delta A\|_z 2^{-\ell z} \lesssim \|\delta A\|_z |t-s|^z$$

from which it is easy to conclude.

Another strategy allows to control the limit over all partitions $\mathcal{P}$ of $[s,t] \subseteq [0,1]$ as their size goes to zero. In this case we let

$$\mathcal{S}_{\mathcal{P}} A = \sum_{[a,b] \in \mathcal{P}} A(a,b)$$

Now fix s,t and for a given partition $\mathcal{P}$ of $[s,t]$ with $n \geqslant 2$ intervals let $[a,b],[b,c]$ two consecutive intervals of $\mathcal{P}$ for which $|c-a| \leqslant r = 2|t-s|/(n-1)$. Note that this pair has to exist otherwise $|t-s| < r(n-1)/2 \leqslant |t-s|$. If we remove the point b from the partition we obtain a partition $\mathcal{P}'$ with $n-1$ intervals and for which

$$|\mathcal{S}_{\mathcal{P}} A - \mathcal{S}_{\mathcal{P}'} A| \lesssim |\delta A(a,b,c)| \lesssim \|\delta A\|_z \frac{2^z}{(n-1)^z} |t-s|^z.$$

Continuing up to get the trivial partition $[s,t]$ we obtain the *maximal estimate*

$$|\mathcal{S}_{\mathcal{P}} A - A(s,t)| \lesssim \|\delta A\|_z \sum_{n \geqslant 1} \frac{2^z}{n^z} |t-s|^z \lesssim \|\delta A\|_z |t-s|^z.$$

By the same reasoning, if $\mathcal{Q} \supseteq \mathcal{P}$ are two partitions then

$$|\mathcal{S}_{\mathcal{Q}} A - \mathcal{S}_{\mathcal{P}} A| \leqslant \sum_{[a,b] \in \mathcal{P}} |\mathcal{S}_{\mathcal{Q} \cap [a,b]} A - A(a,b)| \leqslant \sum_{[a,b] \in \mathcal{P}} \|\delta A\|_z |b-a|^z \lesssim |\mathcal{P}|^{z-1}.$$

This shows that the family $\{\mathcal{S}_{\mathcal{P}} A\}_{\mathcal{P}}$ is Cauchy. If we let $I(t) = \lim_{\mathcal{P}} \mathcal{S}_{\mathcal{P}} A$ where the limit is taken over all partition of $[0,t]$ as the size goes to zero, we have

$$|I(t) - I(s) - A(s,t)| \leqslant \limsup_{\mathcal{P} \subseteq [s,t]} |\mathcal{S}_{\mathcal{P}} A - A(s,t)| \lesssim \|\delta A\|_z |t-s|^z.$$

Riemann sums. The sewing map provides a "correction" to a given 2-increment A in order to make it closed and then exact by the exactness of the complex C_*. Indeed $\delta(A - \Lambda\delta A) = 0$ so there exists a unique $I \in C_1$ such that $\delta I = A - \Lambda\delta A$ and $I(0) = 0$. That this function I is a generalisation of the (Riemann) integral can be understood via the following corollary relating it to limit of Riemann–like sums.

Corollary 2. *Let $\delta A \in C_3^{1+} \cap \delta C_2$ and let $\delta I = A - \Lambda\delta A$, then*

$$S_{\mathcal{P}(s,t)}(A) = \sum_i A(t_i, t_{i+1}) \to I(t,s)$$

where the limit is taken over partitions $\mathcal{P}(s,t) = \{t_i\}_i$ of the interval $[s,t]$ as the size $|\Pi(s,t)|$ of the partition goes to zero.

Proof. Take $z>1$ such that $A\in C_3^z$. Then $|\Lambda\delta A(t_i,t_{i+1})|\lesssim_z |t_{i+1}-t_i|^z\lesssim_z |\mathscr{P}(s,t)|^{z-1}|t_{i+1}-t_i|$ and

$$\sum_i A(t_i,t_{i+1})=\sum_i \delta I(t_i,t_{i+1})+\sum_i \Lambda\delta A(t_i,t_{i+1})=\delta I(s,t)+|\mathscr{P}(s,t)|^{z-1}\sum_i O(|t_{i+1}-t_i|)$$

$$=\delta I(s,t)+|\mathscr{P}(s,t)|^{z-1}O(|t-s|)\to\delta I(s,t)$$

as $|\mathscr{P}(s,t)|\to 0$. □

The Young integral. A first easy consequence of the sewing lemma is the existence of the Young integral.

Theorem 3. *For any $\alpha,\beta\in(0,1)$ such that $\alpha+\beta>1$ the integral map I has a continuous extension $I_Y\colon C^\alpha\times C^\beta\to C^\beta$ such that*

$$|\delta I_Y(f,g)(s,t)-f(s)\delta g(s,t)|\lesssim_{\alpha+\beta}|t-s|^{\alpha+\beta}\|f\|_\alpha\|g\|_\beta$$

and

$$\|I_Y(f,g)\|_{\beta,[a,b]}\leqslant(\|f\|_{\infty,[a,b]}+\|f\|_{\alpha,[a,b]})\|g\|_{\beta,[a,b]}$$

for any $0\leqslant a\leqslant b\leqslant 1$. Moreover

$$I_Y(f,g)(t)=\lim_{|\mathscr{P}(0,t)|\to 0}\sum_{i=0}^{n-1}f(t_i)(g(t_{i+1})-g(t_i))$$

for partitions $\mathscr{P}(0,t)=\{0=t_0\leqslant t_1\leqslant\cdots\leqslant t_n=t\}$ of $[0,t]$.

Proof. Let $A(s,t)=f(s)\delta g(s,t)$ and note that $\delta A(s,u,t)=\delta f(s,u)\delta g(u,t)$ so

$$\|\delta A\|_{\alpha+\beta,[a,b]}\leqslant\|f\|_\alpha\|g\|_{\beta,[a,b]}$$

Since $\alpha+\beta>1$ we can apply the sewing map and let $R_Y(f,g)=\Lambda\delta A$ with $\|R_Y\|_{\alpha+\beta}\lesssim_{\alpha+\beta}\|f\|_\alpha\|g\|_\beta$. Then we let $I_Y(f,g)(t)=A(0,t)-R_Y(0,t)$. Now $\delta I_Y(f,g)(s,t)=A(s,t)-R_Y(s,t)$ and the claim readily follows since for smooth f,g we have $\delta I(f,g)(s,t)=A(s,t)+o_u(|t-s|)$ so in this case $I_Y(f,g)=I(f,g)$. Note moreover that

$$\|I_Y(f,g)\|_{\beta,J}\leqslant\|A\|_{\beta,J}+\|R_Y\|_{\beta,J}\leqslant\|f\|_{\infty,J}\|g\|_{\beta,J}+\|R_Y\|_{\alpha+\beta,J}\leqslant(\|f\|_{\infty,J}+\|f\|_{\alpha,J})\|g\|_{\beta,J}.$$

For the final statement we use Corollary 2. □

An alternative construction of the Young integral. Consider smooth functions f, g and a dyadic decomposition $\{t_i^n\}_{i,n}$ of $[s,t]$. Then

$$I(f,g)(s,t)-f(s)(g(t)-g(s))=\sum_{i=0}^{2^n-1}I(f-f(t_i^n),g)(t_i^n,t_{i+1}^n)-\sum_{k=0}^{n-1}\sum_{i=0}^{2^k-1}H(t_{2i}^{k+1},t_{2i+1}^{k+1},t_{2i+2}^{k+1})$$

where $H(s,u,t)=(f(u)-f(s))(g(t)-g(u))$. So now

$$|I(f,g)(s,t)-f(s)(g(t)-g(s))|\leqslant 2^n|t-s|^{\alpha+1}2^{-(1+\alpha)n}\|f\|_\alpha\|g\|_1+\sum_{k=0}^{n-1}\|f\|_\alpha\|g\|_\beta 2^{k-(\alpha+\beta)(k+1)}$$

$$\leqslant 2^n|t-s|^{\alpha+1}2^{-(1+\alpha)n}\|f\|_\alpha\|g\|_1+\frac{2^{-\alpha-\beta}}{1-2^{1-\alpha-\beta}}\|f\|_\alpha\|g\|_\beta$$

and taking the limit $n\to\infty$ we get

$$|I(f,g)(s,t)-f(s)(g(t)-g(s))|\leqslant\frac{1}{2^{\alpha+\beta}-2}\|f\|_\alpha\|g\|_\beta.$$

In particular this implies that the bilinear map I has the bound

$$\|I(f,g)\|_\beta\leqslant C(\|f\|_\infty+\|f\|_\alpha)\|g\|_\beta \tag{2}$$

so it can be extended continuously to the closure of $C^\alpha\times C^1$ in $C^\alpha\times C^\beta$ with the same bound. Now the closure C_0^β of C^1 in C^β does not coincide C^β but it is enough to observe that $C^\beta\subseteq C_0^{\beta-\varepsilon}$ for any small $\varepsilon>0$. Then if $C^1\ni g_n\to g\in C^\beta$ in C^β we have also $I(f,g_n)\to I(f,g)$ in $C^{\beta-\varepsilon}$ for some small ε. The sequence $(I(f,g_n))_n$ is bounded in C^β and we can extend the bound (2) to all $g\in C^\beta$.

Weighted spaces. To discuss estimates involving the sewing map (for example for rough equations) it is useful to dispose of weighted Hölder norms. Take $\tau>0$ and any function $g\colon\mathbb{R}_+\to\mathbb{R}_+$ positive and non–decreasing and define norms on C_n by

$$\|h\|_{\alpha,\tau,g}:=\sup_{(s_1,\ldots,s_n)\in\Delta_n(s,t)}\frac{|h(s_1,\ldots,s_n)|}{g(s_n/\tau)|s_n-s_1|_\tau^\alpha}<+\infty.$$

where $|a|_\tau=\min(|a|,\tau)$. Note that we have the relations

$$\|h\|_{\alpha,\tau,g}\leqslant\tau^{\beta-\alpha}\|h\|_{\beta,\tau,g},\qquad\beta\geqslant\alpha$$

and for $f\in C_1$

$$\|f\|_{\infty,\tau,g}:=\sup_{t\geqslant 0}\frac{|f(t)|}{g(t/\tau)}\leqslant\sup_{t\geqslant 0}\frac{|f(0)|}{g(t/\tau)}+\tau^\alpha\,\|\delta f\|_{\alpha,\tau,g}.$$

The next corollary gives estimates of the sewing map in these weighted norms.

Corollary 4. *For any $z>1$*

$$\|\Lambda\delta A\|_{z,\tau,G_\tau}\leqslant\frac{2^z}{2^z-2}\|\delta A\|_{z,\tau,g}$$

where

$$G(t)=g(t)+\int_0^t g(s)\mathrm{d}s.$$

Proof. Given $0 \leqslant s < t$ and $n \geq 1$, set $t_i^n = s + i\,2^{-n}(t-s)$, and consider

$$\Lambda\delta A(s,t) = \sum_{i=0}^{2^n-1} \Lambda\delta A(t_i^n, t_{i+1}^n) \; - A(s,t) = \sum_{k=0}^{n-1} \sum_{i=0}^{2^k-1} \delta A(t_{2i}^{k+1}, t_{2i+1}^{k+1}, t_{2i+2}^{k+1}).$$

We already know that $|\Lambda\delta A(s,t)| \leqslant C_z \|\delta A\|_{z,[s,t]} |t-s|^z$ with $C_z = \frac{2^z}{2^z - 2}$. Now, if $|t-s| \leqslant \tau$, we have $|t-s|_\tau = |t-s|$, so the above inequality implies clearly that we have

$$\frac{|\Lambda\delta A(s,t)|}{G_\tau(t/\tau)\;|t-s|_\tau^z} \leqslant \frac{|\Lambda\delta A(s,t)|}{g(t/\tau)\;|t-s|_\tau^z} \leqslant C_z \frac{\|\delta A\|_{z,[s,t]}}{g(t/\tau)} \leqslant C_z \|\delta A\|_{z,\tau,g} \tag{3}$$

in that case. If now $|t-s| > \tau$, divide the interval $[s,t]$ into sufficiently many intervals $[t_k, t_{k+1}]$ of equal length no greater than τ, say N sub-intervals. Since $\delta\Lambda\delta A = \delta A$ we have

$$(\Lambda\delta A)(s,t) = \sum_{k=0}^{N-1} (\Lambda\delta A)(t_k, t_{k+1}). + \sum_{k=0}^{N-1} \delta A(s, t_k, t_{k+1}).$$

The choice of times t_k and inequality (3) guarantee that

$$\sum_{k=0}^{N-1} |(\Lambda\delta A)(t_k, t_{k+1})| \leqslant C_z \|\delta A\|_{z,\tau,g}\, \tau^z \sum_{k=0}^{N-1} g(t_{k+1}/\tau) \leqslant C_z \|\delta A\|_{z,\tau,g}\, |t-s|_\tau^z G(t/\tau)$$

where we used that

$$\sum_{k=0}^{N-1} g(t_{k+1}/\tau) \leqslant \sum_{k=0}^{N-1} \frac{1}{\tau} \int_{t_{k+1}}^{t_{k+2}\wedge t} g(s/\tau) \mathrm{d}s \leqslant \int_0^t g(s/\tau) \frac{\mathrm{d}s}{\tau} = \int_0^{t/\tau} g(s) \mathrm{d}s \leqslant G(t/\tau).$$

The conclusion follows from the other elementary inequality

$$\sum_{k=0}^{N-1} |\delta A(s, t_k, t_{k+1})| \leqslant \tau^z \|\delta A\|_{z,\tau,g} \sum_{k=0}^{N-1} g(t_{k+1}/\tau) \leqslant \|\delta A\|_{z,\tau,g}\, \tau^z G(t/\tau).$$

$\square$

Two useful choices for g are :

a) $g(t) = \exp(t)$ in which case we denote the correspondig weighted norm by $\|\cdot\|_{\alpha,\tau,\exp}$. We have $G = 2\exp(t)$ so $\|\cdot\|_{\alpha,\tau,\exp} = 2\,\|\cdot\|_{\alpha,\tau,G}$.

b) $g(t) = 1$ with norm denoted by $\|\cdot\|_{\alpha,\tau}$ and for which $G(t) = 1 + t$ so that again if the set of times is bounded by T we have, if $\|\cdot\|_{\alpha,\tau} \leqslant (1+T)\,\|\cdot\|_{\alpha,\tau,G}$.