Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces. II.

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This is an addendum to the paper "Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces", JFA 259 (2010), 28-56, by K. Bacher and the second author. We prove the tensorization property for the curvature-dimension condition, add some detailed calculations – including explicit dependence of constants – and comment on assumptions and conjectures concerning the local-to-global statement in [1] and [6], resp.

1 Tensorization property of the curvature-dimension condition

Theorem 1.1. *Let (M_i, d_i, m_i) be non-branching metric measure spaces satisfying the curvature-dimension $CD(K_i, N_i)$ with $N_i \geq 1$ for $i = 1, 2, \cdots, k$. Then*

$$(M, d, m) := \bigotimes_{i=1}^{k} (M_i, d_i, m_i)$$

satisfies $CD(\min_i K_i, \sum_{i=1}^{k} N_i)$.

The proof of this result essentially depends on the estimate in the following Lemma. The latter was already obtained by S. Ohta (see [3] Claim 3.4) with a long computation. Below we present a short proof based on Lemma 1.2 in [5]. The analogous estimate with the coefficients $\tau_{K,N}^{(t)}$ replaced by the slightly smaller coefficients $\sigma_{K,N}^{(t)}$ had been used in [1] to deduce the tensorization property of the *reduced* curvature-dimension condition.

Lemma 1.2. *For any $K, K' \in \mathbb{R}$, any $N, N' \in (1, \infty)$, any $t \in [0, 1]$ and any $\theta_1, \theta_2 \in \mathbb{R}_+$ with $\theta^2 = \theta_1^2 + \theta_2^2$ we have*

$$\tau_{K,N}^{(t)}(\theta_1)^N \cdot \tau_{K,N'}^{(t)}(\theta_2)^{N'} \geq \tau_{K,N+N'}^{(t)}(\theta)^{N+N'}$$

Proof. The inequality

$$\sigma_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \sigma_{K+K',N+N'}^{(t)}(\theta)^{N+N'}.$$

derived in [5], Lemma 1.2, implies

$$\tau_{K',N'}^{(t)}(\theta)^{N'} = t \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} = \sigma_{0,1}^{(t)}(\theta)^1 \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} \geq \sigma_{K',N'}^{(t)}(\theta)^{N'}.$$

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Combining this with another inequality from [5], Lemma 1.2:

$$\tau_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \tau_{K+K',N+N'}^{(t)}(\theta)^{N+N'}$$

yields

$$\tau_{K,N}^{(t)}(\theta)^N \cdot \tau_{K',N'}^{(t)}(\theta)^{N'} \geq \tau_{K+K',N+N'}^{(t)}(\theta)^{N+N'}. \tag{1.1}$$

Now observe that $\tau_{K,N}^{(t)}(\theta_1) = \tau_{\theta_1^2 K/\theta^2,N}^{(t)}(\theta)$ and $\tau_{K,N'}^{(t)}(\theta_2) = \tau_{\theta_2^2 K/\theta^2,N'}^{(t)}(\theta)$. Then the claim follows from (1.1). □

Lemma 1.3. *Let a, b, c, d be positive numbers and $p \in (0, 1)$, then*

$$a^p b^{1-p} + c^p d^{1-p} \leq (a + c)^p (b + d)^{1-p}.$$

Proof. By the concavity of the function $\ln x$, we have

$$p \ln \frac{a}{a+c} + (1-p) \ln \frac{b}{b+d} \leq \ln \left(p \cdot \frac{a}{a+c} + (1-p) \cdot \frac{b}{b+d} \right)$$

which is equavilent to

$$\left(\frac{a}{a+c}\right)^p \left(\frac{b}{b+d}\right)^{1-p} \leq p \cdot \frac{a}{a+c} + (1-p) \cdot \frac{b}{b+d} \tag{1.2}$$

Similarly, we have

$$\left(\frac{c}{a+c}\right)^p \left(\frac{d}{b+d}\right)^{1-p} \leq p \cdot \frac{c}{a+c} + (1-p) \cdot \frac{d}{b+d} \tag{1.3}$$

Combine (1.2) and (1.3), we obtain

$$\left(\frac{a}{a+c}\right)^p \left(\frac{b}{b+d}\right)^{1-p} + \left(\frac{c}{a+c}\right)^p \left(\frac{d}{b+d}\right)^{1-p} \leq 1.$$

In other words,

$$a^p b^{1-p} + c^p d^{1-p} \leq (a + c)^p (b + d)^{1-p}.$$

□

Proof of Theorem 1.1. We basically follow the argument in [1] and so we only sketch the main steps. Please see [1] for more details.

Step 1: Without loss of generality, we assume $k = 2$. And we can assume $K_1 = K_2 = K$ due to the fact that $CD(K_1, N)$ implies $CD(K_2, N)$ if $K_1 \geq K_2$.

Step 2: Consider the special case where ν_0 and ν_1 are product measures. In this step, we only need to replace σ by τ on page 43 in [1]. In the following, we write down the formula corresponding to [1].

$$
\begin{aligned}
&\tau^{(1-t)}_{K,N_1+N_2}(d(x_0,x_1))\rho_0(x_0)^{-1/(N_1+N_2)} + \tau^{(t)}_{K,N_1+N_2}(d(x_0,x_1))\rho_1(x_1)^{-1/(N_1+N_2)}\\
=&\tau^{(1-t)}_{K,N_1+N_2}(d(x_0,x_1))\rho_0^{(1)}(x_0^{(1)})^{-1/(N_1+N_2)}\rho_0^{(2)}(x_0^{(2)})^{-1/(N_1+N_2)}\\
&+\tau^{(t)}_{K,N_1+N_2}(d(x_0,x_1))\rho_1^{(1)}(x_1^{(1)})^{-1/(N_1+N_2)}\rho_1^{(2)}(x_1^{(2)})^{-1/(N_1+N_2)}\\
\leq&\prod_{i=1}^{2}\tau^{(1-t)}_{K,N_i}(d_i(x_0^{(i)},x_1^{(i)}))^{N_i/(N_1+N_2)}\rho_0^{(i)}(x_0^{(i)})^{-1/(N_1+N_2)}\\
&+\prod_{i=1}^{2}\tau^{(t)}_{K,N_i}(d_i(x_0^{(i)},x_1^{(i)}))^{N_i/(N_1+N_2)}\rho_1^{(i)}(x_1^{(i)})^{-1/(N_1+N_2)}\\
\leq&\prod_{i=1}^{2}\left[\tau^{(1-t)}_{K,N_i}(d_i(x_0^{(i)},x_1^{(i)}))\rho_0^{(i)}(x_0^{(i)})^{-1/N_i}+\tau^{(t)}_{K,N_i}(d_i(x_0^{(i)},x_1^{(i)}))\rho_1^{(i)}(x_1^{(i)})^{-1/N_i}\right]^{N_i/(N_1+N_2)}\\
\leq&\prod_{i=1}^{2}\rho_t^{(i)}(\gamma_t^{(i)}(x_0^{(i)},x_1^{(i)}))^{-1/(N_1+N_2)}\\
=&\rho_t(\gamma_t(x_0,x_1))^{-1/(N_1+N_2)}.
\end{aligned}
$$

The first inequality follows from Lemma 1.2. The second inequality follows from Lemma 1.3. The third inequality follows from the definition of curvature-dimension condition.

Step 3: For general case, we approximate ν_0 and ν_1 by the average of mutually singular product probability measures $\nu_{0,n}$ and $\nu_{1,n}$ as in [1], where $n = 1, 2, \cdots$. Then we obtain geodesics γ_n of $\nu_{0,n}$ and $\nu_{1,n}$, and passing some subsequence, we obtain a geodesic γ of ν_0 and ν_1 satisfying the curvature-dimension condition by using the lower-semicontinuity of the Rényi entropy. Then we conclude that

$$(M,d,m) := \bigotimes_{i=1}^{2}(M_i,d_i,m_i)$$

satisfies $CD(K, N_1 + N_2)$. $\square$

2 Details to the proof of Proposition 5.5 in [1]

The proof of Proposition 5.5 in [1] uses the following fact (with $\tilde{K}, N', N$ in the place of K, N, N_0).

Lemma 2.1. *For each $N_0 > 1$ and for each pair $K > K'$ there exists a $\theta^* > 0$ s.t. for all $\theta \in (0, \theta^*)$, all $t \in (0, 1)$ and all $N \in [N_0, \infty)$*

$$\tau^{(t)}_{K',N}(\theta) \leq \sigma^{(t)}_{K,N}(\theta). \tag{2.1}$$

The proof of this fact presented in the above mentioned paper contains some sketchy and incomplete arguments (in particular, concerning the uniform dependence of the constants in the regime $t \nearrow 1$). We will present a detailed proof below. To simplify notation, however, in our presentation we will restrict ourselves to the case $K > K' > 0$. Recall that in this case

$$\sigma^{(t)}_{K,N}(\theta) = \frac{\sin\left(\sqrt{\frac{K}{N}}t\theta\right)}{\sin\left(\sqrt{\frac{K}{N}}\theta\right)} \quad \text{and} \quad \tau^{(t)}_{K,N}(\theta) = t^{1/N}\cdot\sigma^{(t)}_{K,N-1}(\theta)^{1-1/N}.$$

In the other cases $0 > K > K'$ and $K > 0 > K'$ completely similar arguments will apply.

Claim 2.2. *$\exists C_0, \theta_0 :\ \forall t \in (0,1), \forall \theta \in (0, \theta_0)$:*

$$\frac{\sin(t\theta)}{\sin(\theta)} \le t \cdot \left(1 + \frac{1}{6}(1-t^2)\theta^2 \cdot \left[1 + C_0\theta^2\right]\right)$$

and

$$\frac{\sin(t\theta)}{\sin(\theta)} \ge t \cdot \left(1 + \frac{1}{6}(1-t^2)\theta^2 \cdot \left[1 - C_0\theta^2\right]\right).$$

Proof. Uniformly in $t \in (0, \frac{1}{2}]$, the claim immediately follows from the straightforward asymptotics

$$\frac{\sin(t\theta)}{\sin(\theta)} = \frac{t\theta - \frac{1}{6}t^3\theta^3 + O(\theta^5)}{\theta - \frac{1}{6}\theta^3 + O(\theta^5)} = t \cdot \left(1 + \frac{1}{6}(1-t^2)\theta^2 + O(\theta^4)\right) \qquad \text{for } \theta \to 0$$

already presented in the proof of Proposition 5.5. For $t \in [\frac{1}{2}, 1)$ we use this asymptotics (with $1-t$ in the place of t) to deduce

$$\begin{aligned}
\frac{\sin(t\theta)}{\sin(\theta)} &= \cos((1-t)\theta) - \cos(\theta)\frac{\sin((1-t)\theta)}{\sin(\theta)} \\
&= \left[1 - \frac{(1-t)^2}{2}\theta^2 + (1-t)\cdot O(\theta^4)\right] - \left[1 - \frac{1}{2}\theta^2 + O(\theta^4)\right](1-t)\left(1 + \frac{2t(1-t)}{6}\theta^2 + O(\theta^4)\right) \\
&= t \cdot \left(1 + \frac{1}{6}(1-t^2)\theta^2 \cdot \left[1 + O(\theta^2)\right]\right).
\end{aligned}$$

□

Claim 2.3. *Put $\theta_1 = \min\{\theta_0 \frac{1}{\sqrt{K}}, \frac{1}{\sqrt{C_0 K}}\}$. Then for all $\theta \in (0, \theta_1)$, all $t \in (0,1)$ and all $N \in [1, \infty)$*

$$\sigma_{K,N}^{(t)}(\theta)^N \ge t^N \cdot \left(1 + \frac{1}{6}(1-t^2)K\theta^2 \cdot \left[1 - C_0 K\theta^2\right]\right). \tag{2.2}$$

Proof. According to Claim 1.2 (now with $\sqrt{\frac{K}{N}}\theta$ in the place of θ) and using the fact that $1 + \epsilon \le (1 + \epsilon/N)^N$ we obtain

$$\begin{aligned}
\sigma_{K,N}^{(t)}(\theta)^N &\ge t^N \cdot \left(1 + \frac{1}{6}(1-t^2)\frac{K}{N}\theta^2 \cdot \left[1 - C_0\frac{K}{N}\theta^2\right]\right)^N \\
&\ge t^N \cdot \left(1 + \frac{1}{6}(1-t^2)K\theta^2 \cdot \left[1 - C_0 K\theta^2\right]\right).
\end{aligned}$$

□

Claim 2.4. *Put $C_1 = \frac{C_0}{N_0 - 1} + \frac{1}{3}$ and $\theta_2 = \min\{\theta_0 \frac{N_0-1}{\sqrt{K}}, \sqrt{\frac{8}{K(1+C_0\theta_0^2)}}\}$. Then for all $\theta \in (0, \theta_2)$, all $t \in (0,1)$ and all $N \in [N_0, \infty)$*

$$\tau_{K,N}^{(t)}(\theta)^N \le t^N \cdot \left(1 + \frac{1}{6}(1-t^2)K\theta^2 \cdot \left[1 + C_1 K\theta^2\right]\right). \tag{2.3}$$

Proof. Note that $(1 + \frac{\epsilon}{N-1})^{N-1} \le e^\epsilon \le 1 + \epsilon + \epsilon^2$ for all $\epsilon \in (0, \frac{1}{3})$ and all $N \ge N_0 > 1$. Hence, Claim 1.2 implies

$$\begin{aligned}
\tau_{K,N}^{(t)}(\theta)^N &\le t^N \cdot \left(1 + \frac{1}{6}(1-t^2)\frac{K}{N-1}\theta^2 \cdot \left[1 + C_0\frac{K}{N-1}\theta^2\right]\right)^{N-1} \\
&\le t^N \cdot \left(1 + \frac{1}{6}(1-t^2)K\theta^2 \cdot \left[1 + C_1 K\theta^2\right]\right).
\end{aligned}$$

□

Now choose $\theta^* \le \min\{\theta_1, \theta_2\}$ and such that $[C_0 K^2 + C_1 {K'}^2](\theta^*)^2 \le K - K'$. Then Claim 1.4 (with K' in the place of K) and Claim 1.3 imply

$$\sigma_{K,N}^{(t)}(\theta)^N - \tau_{K',N}^{(t)}(\theta)^N \ge t^N \frac{1}{6}(1-t^2)\theta^2 \left(K\left[1 - C_0 K\theta^2\right] - K'\left[1 + C_1 K'\theta^2\right]\right) \ge 0$$

which completes the *proof of Lemma 1.1.*

3 Disproving Conjecture 30.34 in [6]

Cédric Villani in his monograph [6] formulated a conjecture which – if it were true – would allow him to prove the local-to-global property for $CD(K,N)$ (Theorem 30.37). We will prove that *this conjecture is false.*

In our terminology, it reads as follows.

Conjecture. *Given $N > 1$, $K \in \mathbb{R} \setminus \{0\}$ and $f : [0,L] \to \mathbb{R}$ with $L \le \pi\sqrt{\frac{N-1}{K}}$ provided $K > 0$ and arbitrary $L \in \mathbb{R}_+$ otherwise. If*

$$f((1-t)\theta_0 + t\theta_1) \ge \tau_{K,N}^{(1-t)}(|\theta_0 - \theta_1|) \cdot f(\theta_0) + \tau_{K,N}^{(t)}(|\theta_0 - \theta_1|) \cdot f(\theta_1) \tag{3.1}$$

holds true for all $t \in (0,1)$ and all $\theta_0, \theta_1 \in [0,L]$ with $|\theta_0 - \theta_1|$ small then it holds true for all $t \in (0,1)$ and all $\theta_0, \theta_1 \in [0,L]$.

In order to construct a **counterexample**, in the case $K > 0$ choose $\tilde{K} > K$ such that

$$\cos\left(\frac{L}{2}\sqrt{\frac{\tilde{K}}{N}}\right) > \cos\left(\frac{L}{2}\sqrt{\frac{K}{N-1}}\right)^{1-1/N}.$$

Note that such a $\tilde{K}$ exists since

$$\cos\left(\frac{L}{2}\sqrt{\frac{K}{N}}\right) > \cos\left(\frac{L}{2}\sqrt{\frac{K}{N-1}}\right)^{1-1/N}$$

which in turn is equivalent to $\sigma_{K,N}^{(1/2)}(L) < \tau_{K,N}^{(1/2)}(L)$, the latter being a general fact, derived in [5], Lemma 1.2. In the case $K < 0$, the same argument allows to choose $\tilde{K} \in (K, 0)$ such that $\cosh\left(\frac{L}{2}\sqrt{\frac{-\tilde{K}}{N}}\right) > \cosh\left(\frac{L}{2}\sqrt{\frac{-K}{N-1}}\right)^{1-1/N}$.

Let $f : [0,L] \to \mathbb{R}$ be any positive solution to the ODE $f'' = -\frac{\tilde{K}}{N} \cdot f$. Then

$$f((1-t)\theta_0 + t\theta_1) = \sigma_{\tilde{K},N}^{(1-t)}(|\theta_1 - \theta_0|) \cdot f(\theta_0) + \sigma_{\tilde{K},N}^{(t)}(|\theta_1 - \theta_0|) \cdot f(\theta_1)$$

for all $t \in (0,1)$ and all $\theta_0, \theta_1 \in [0,L]$. Hence, according to Lemma 1.1 for $|\theta_0 - \theta_1|$ being sufficiently small

$$f((1-t)\theta_0 + t\theta_1) \ge \tau_{K,N}^{(1-t)}(|\theta_1 - \theta_0|) \cdot f(\theta_0) + \tau_{K,N}^{(t)}(|\theta_1 - \theta_0|) \cdot f(\theta_1).$$

If the Conjecture were true it would then for instance imply

$$f(L/2) \ge \tau_{K,N}^{(1/2)}(L) \cdot [f(0) + f(L)] = \frac{1}{\cos\left(\frac{L}{2}\sqrt{\frac{K}{N-1}}\right)^{1-1/N}} \cdot \frac{f(0)+f(L)}{2},$$

with appropriate interpretation of the denominator of the RHS in the case $K < 0$. Now let us make a specific choice for f, namely, $f(\theta) = \cos\left((\theta - \frac{L}{2})\sqrt{\frac{\tilde{K}}{N}}\right)$. Then the previous inequality reads as follows

$$1 \geq \frac{\cos\left(\frac{L}{2}\sqrt{\frac{\tilde{K}}{N}}\right)}{\cos\left(\frac{L}{2}\sqrt{\frac{K}{N-1}}\right)^{1-1/N}}$$

which is a contradiction. $\square$

Remark. Let us emphasize that the above counterexample is *not* a counterexample to the local-to-global property of the curvature-dimension condition $CD(K, N)$. It merely says that the way proposed in [6], Theorem 30.37, to prove this local-to-global property will not work.

In the nontrivial case $K/N \neq 0$, it is still an open problem whether the local version of the curvature-dimension condition $CD(K, N)$ implies the corresponding global version.

As one of the main results in [1], the local-to-global property for the reduced curvature-dimension condition $CD^*(K, N)$ was proven for all pairs of K and N. Moreover, it was shown that the local versions of $CD(K, N)$ and $CD^*(K, N)$ are equivalent. Hence, the remaining challenge is

either prove or disprove that $CD_{loc}(K, N)$ implies $CD(K, N)$

or equivalently

either prove or disprove that $CD^*(K, N)$ implies $CD(K, N)$.

4 A Remark concerning $\mathcal{P}_\infty(M, d, m)$ being a Geodesic Space

In Theorem 5.1 of the afore mentioned paper [1], we had assumed that $\mathcal{P}_\infty(M, d, m)$ is a geodesic space. *This assumption can equivalently be replaced by the much simpler assumption that* supp$[m]$ *is a geodesic space.* The latter always follows from the preceding (cf. Remark 4.18(ii) in [4]). The converse implication holds true under the assumption of $CD^*_{loc}(K, N)$ for some finite N.

Indeed, this implies $CD_{loc}(K-, N)$ with "CD" being defined in the sense of [5]. Due to the non-branching assumption this is equivalent to an analogous "CD" definition in the sense of [2] (Theorem 30.32 in [6] and/or Proposition 4.2 in [5]). The latter in turn implies that $\mathcal{P}_\infty(M, d, m)$ is a geodesic space provided M is geodesic with full support (Theorem 30.19(ii) in [6], cf. also proof of Theorem 30.37) or at least if supp$[m]$ is a geodesic space.

In Theorem 7.10 of [1] the assumption that m has full support has to be added. Then $\hat{M}$ is a geodesic space with full support and the result of Theorem 5.1 applies.

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