Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces.

II.

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This is an addendum to the paper "Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces", JFA 259 (2010), 28-56, by K. Bacher and the second author. We prove the tensorization property for the curvature-dimension condition, add some detailed calculations – including explicit dependence of constants – and comment on assumptions and conjectures concerning the local-to-global statement in [1] and [6], resp.

1 Tensorization property of the curvature-dimension condition

Theorem 1.1. Let \((M_i, d_i, m_i)\) be non-branching metric measure spaces satisfying the curvature-dimension \(CD(K_i, N_i)\) with \(N_i \geq 1\) for \(i = 1, 2, \ldots, k\). Then

\[(M, d, m) := \bigotimes_{i=1}^k (M_i, d_i, m_i)\]

satisfies \(CD(\min_i K_i, \sum_{i=1}^k N_i)\).

The proof of this result essentially depends on the estimate in the following Lemma. The latter was already obtained by S. Ohta (see [3] Claim 3.4) with a long computation. Below we present a short proof based on Lemma 1.2 in [5]. The analogous estimate with the coefficients \(\tau_{K,N}^{(t)}\) replaced by the slightly smaller coefficients \(\sigma_{K,N}^{(t)}\) had been used in [1] to deduce the tensorization property of the reduced curvature-dimension condition.

Lemma 1.2. For any \(K, K' \in \mathbb{R}\), any \(N, N' \in (1, \infty)\), any \(t \in [0, 1]\) and any \(\theta_1, \theta_2 \in \mathbb{R}^+\) with \(\theta^2 = \theta_1^2 + \theta_2^2\) we have

\[\tau_{K,N}(\theta_1)^N \cdot \tau_{K',N'}(\theta_2)^{N'} \geq \tau_{K,N+N'}^{(t)}(\theta)^{N+N'}\]

Proof. The inequality

\[\sigma_{K,N}(\theta)^N \cdot \sigma_{K',N'}(\theta)^{N'} \geq \sigma_{K+K',N+N'}^{(t)}(\theta)^{N+N'}\]

derived in [5], Lemma 1.2, implies

\[\tau_{K',N'}^{(t)}(\theta)^{N'} = t \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} = \sigma_{0,1}^{(t)}(\theta)^1 \cdot \sigma_{K',N'-1}^{(t)}(\theta)^{N'-1} \geq \sigma_{K',N'}^{(t)}(\theta)^{N'}\]

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Combining this with another inequality from [5], Lemma 1.2:

\[ \tau_{K,N}(\theta)^N \cdot \sigma_{K',N'}(\theta)^N \geq \tau_{K+K',N+N'}^{(t)}(\theta)^{N+N'} \]

yields

\[ \tau_{K,N}(\theta)^N \cdot \tau_{K',N'}^{(t)}(\theta)^N \geq \tau_{K+K',N+N'}^{(t)}(\theta)^{N+N'}. \]  \hspace{1cm} (1.1)

Now observe that \[ \tau_{K,N}(\theta_1) = \tau_{\theta_0 K/\theta_2 N}(\theta) \] and \[ \tau_{K,N}(\theta_2) = \tau_{\theta_0 K/\theta_2 N'}(\theta). \] Then the claim follows from (1.1).

**Lemma 1.3.** Let \( a, b, c, d \) be positive numbers and \( p \in (0,1) \), then

\[ a^pb^{1-p} + c^pd^{1-p} \leq (a + c)^p(b + d)^{1-p}. \]

**Proof.** By the concavity of the function \( \ln x \), we have

\[ p \ln \frac{a}{a + c} + (1 - p) \ln \frac{b}{b + d} \leq \ln \left( p \cdot \frac{a}{a + c} + (1 - p) \cdot \frac{b}{b + d} \right) \]

which is equivalent to

\[ \left( \frac{a}{a + c} \right)^p \left( \frac{b}{b + d} \right)^{1-p} \leq \frac{a}{a + c} \cdot \frac{b}{b + d} \]  \hspace{1cm} (1.2)

Similarly, we have

\[ \left( \frac{c}{a + c} \right)^p \left( \frac{d}{b + d} \right)^{1-p} \leq \frac{c}{a + c} \cdot \frac{d}{b + d} \]  \hspace{1cm} (1.3)

Combine (1.2) and (1.3), we obtain

\[ \left( \frac{a}{a + c} \right)^p \left( \frac{b}{b + d} \right)^{1-p} + \left( \frac{c}{a + c} \right)^p \left( \frac{d}{b + d} \right)^{1-p} \leq 1. \]

In other words,

\[ a^pb^{1-p} + c^pd^{1-p} \leq (a + c)^p(b + d)^{1-p}. \]

**Proof of Theorem 1.1.** We basically follow the argument in [1] and so we only sketch the main steps. Please see [1] for more details.

**Step 1:** Without loss of generality, we assume \( k = 2 \). And we can assume \( K_1 = K_2 = K \) due to the fact that \( CD(K_1, N) \) implies \( CD(K_2, N) \) if \( K_1 \geq K_2 \).

**Step 2:** Consider the special case where \( \nu_0 \) and \( \nu_1 \) are product measures. In this step, we only need to replace \( \sigma \) by \( \tau \) on page 43 in [1]. In the following, we write down the formula corresponding
The proof of Proposition 5.5 in [1] uses the following fact (with 

details to the proof of Proposition 5.5 in [1].

The third inequality follows from the definition of curvature-dimension condition.

Step 3: For general case, we approximate \( \nu_0 \) and \( \nu_1 \) by the average of mutually singular product probability measures \( \nu_{0,n} \) and \( \nu_{1,n} \) as in [1], where \( n = 1, 2, \cdots \). Then we obtain geodesics \( \gamma_n \) of \( \nu_{0,n} \) and \( \nu_{1,n} \), and passing some subsequence, we obtain a geodesic \( \gamma \) of \( \nu_0 \) and \( \nu_1 \) satisfying the curvature-dimension condition by using the lower-semicontinuity of the R\( \ddot{e} \)nyi entropy. Then we conclude that

\[
(M, d, m) := \bigotimes_{i=1}^2 (M_i, d_i, m_i)
\]
satisfies \( CD(K, N_1 + N_2) \).

2 Details to the proof of Proposition 5.5 in [1]

The proof of Proposition 5.5 in [1] uses the following fact (with \( \hat{K}, N', N \) in the place of \( K, N, N_0 \)).

Lemma 2.1. For each \( N_0 > 1 \) and for each pair \( K > K' \) there exists a \( \theta^* > 0 \) s.t. for all \( \theta \in (0, \theta^*) \), all \( t \in (0, 1) \) and all \( N \in [N_0, \infty) \)

\[
\tau_{K',N}(\theta) \leq \sigma_{K,N}(\theta).
\]

The proof of this fact presented in the above mentioned paper contains some sketchy and incomplete arguments (in particular, concerning the uniform dependence of the constants in the regime \( t \not\to 1 \)). We will present a detailed proof below. To simplify notation, however, in our presentation we will restrict ourselves to the case \( K > K' > 0 \). Recall that in this case

\[
\sigma_{K,N}(\theta) = \frac{\sin \left( \sqrt{\frac{K}{N}} \theta^t \right)}{\sin \left( \sqrt{\frac{K'}{N}} \theta^t \right)} \quad \text{and} \quad \tau_{K,N}(\theta) = t^{1/N} \sigma_{K,N-1}(\theta)^{1-1/N}.
\]

In the other cases \( 0 > K > K' \) and \( K > 0 > K' \) completely similar arguments will apply.
Claim 2.2. For all $t \in (0, 1)$ and all $N \in [1, \infty)$, we have
\[
\frac{\sin(t \theta)}{\sin(\theta)} \leq t \left( 1 + \frac{1}{6} (1 - t^2) \theta^2 \cdot [1 + C_0 \theta^2] \right)
\]
and
\[
\frac{\sin(t \theta)}{\sin(\theta)} \geq t \left( 1 + \frac{1}{6} (1 - t^2) \theta^2 \cdot [1 - C_0 \theta^2] \right).
\]

Proof. Uniformly in $t \in (0, \frac{1}{4})$, the claim immediately follows from the straightforward asymptotics
\[
\frac{\sin(t \theta)}{\sin(\theta)} = \frac{t \theta - \frac{1}{6} t^3 \theta^3 + O(\theta^4)}{\theta - \frac{1}{6} \theta^3 + O(\theta^4)} = t \left( 1 + \frac{1}{6} (1 - t^2) \theta^2 + O(\theta^4) \right)
\]
for $\theta \to 0$ already presented in the proof of Proposition 5.5. For $t \in [\frac{1}{2}, 1)$ we use this asymptotics (with $1 - t$ in the place of $t$) to deduce
\[
\frac{\sin(t \theta)}{\sin(\theta)} = \frac{\cos((1 - t) \theta) - \cos(\theta) \sin((1 - t) \theta)}{\sin(\theta)} = \left[ 1 - \frac{(1 - t)^2}{2} \theta^2 + (1 - t) \cdot O(\theta^4) \right] - \left[ 1 - \frac{1}{2} \theta^2 + O(\theta^4) \right] (1 - t) \left( 1 + \frac{2t(1 - t)}{6} \theta^2 + O(\theta^4) \right)
\]
\[
= t \left( 1 + \frac{1}{6} (1 - t^2) \theta^2 \cdot [1 + O(\theta^2)] \right).
\]

\[
\square
\]

Claim 2.3. Put $\theta_1 = \min\{\theta_1 \frac{1}{\sqrt{K}}, \frac{1}{\sqrt{K/C_0}}\}$. Then for all $\theta \in (0, \theta_1)$, all $t \in (0, 1)$ and all $N \in [1, \infty)$
\[
\sigma_{K,N}(\theta)^N \geq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) K \theta^2 \cdot [1 - C_0 K \theta^2] \right).
\]

Proof. According to Claim 1.2 (now with $\sqrt{K/N}$ in the place of $\theta$) and using the fact that $1 + \epsilon \leq (1 + \epsilon/N)^N$ we obtain
\[
\sigma_{K,N}(\theta)^N \geq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) \frac{K}{N} \theta^2 \cdot [1 - C_0 \frac{K}{N} \theta^2] \right)^N
\]
\[
\geq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) K \theta^2 \cdot [1 - C_0 K \theta^2] \right).
\]

\[
\square
\]

Claim 2.4. Put $C_1 = \frac{C_0}{N_0 - 1} + \frac{1}{3}$ and $\theta_2 = \min\{\theta_2 \frac{N_0 - 1}{\sqrt{K}}, \frac{8}{\sqrt{K(1 + C_0 \theta^2)}}\}$. Then for all $\theta \in (0, \theta_2)$, all $t \in (0, 1)$ and all $N \in [N_0, \infty)$
\[
\tau_{K,N}(\theta)^N \leq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) K \theta^2 \cdot [1 + C_1 K \theta^2] \right).
\]

Proof. Note that $(1 + \frac{\epsilon}{N - 1})^{N-1} \leq e^\epsilon \leq 1 + \epsilon + \epsilon^2$ for all $\epsilon \in (0, \frac{1}{4})$ and all $N \leq N_0 > 1$. Hence, Claim 1.2 implies
\[
\tau_{K,N}(\theta)^N \leq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) \frac{K}{N - 1} \theta^2 \cdot [1 + C_0 \frac{K}{N - 1} \theta^2] \right)^{N-1}
\]
\[
\leq t^N \cdot \left( 1 + \frac{1}{6} (1 - t^2) K \theta^2 \cdot [1 + C_1 K \theta^2] \right).
\]

\[
\square
\]
Now choose \( \theta^* \leq \min\{\theta_1, \theta_2\} \) and such that \( [C_0 K^2 + C_1 K'^2](\theta^*)^2 \leq K - K' \). Then Claim 1.4 (with \( K' \) in the place of \( K \)) and Claim 1.3 imply

\[
\sigma_{K,N}^{(t)}(\theta)^N - \tau_{K',N}^{(t)}(\theta)^N \geq t^N \frac{1}{6}(1 - t^2)\theta^2 (K [1 - C_0 K\theta^2] - K' [1 + C_1 K'\theta^2]) \geq 0
\]

which completes the proof of Lemma 1.1.

### 3 Disproving Conjecture 30.34 in [6]

Cédric Villani in his monograph [6] formulated a conjecture which – if it were true – would allow him to prove the local-to-global property for \( CD(K, N) \) (Theorem 30.37). We will prove that this conjecture is false.

In our terminology, it reads as follows.

**Conjecture.** Given \( N > 1 \), \( K \in \mathbb{R} \setminus \{0\} \) and \( f : [0, L] \to \mathbb{R} \) with \( L \leq \pi \sqrt{\frac{N-1}{K}} \) provided \( K > 0 \) and arbitrary \( L \in \mathbb{R}_+ \) otherwise. If

\[
f((1 - t)\theta_0 + t\theta_1) \geq \tau_{K,N}^{(1-t)}(\theta_0 - \theta_1) \cdot f(\theta_0) + \tau_{K,N}^{(t)}(\theta_0 - \theta_1) \cdot f(\theta_1)
\]

holds true for all \( t \in (0, 1) \) and all \( \theta_0, \theta_1 \in [0, L] \) with \( |\theta_0 - \theta_1| \) small then it holds true for all \( t \in (0, 1) \) and all \( \theta_0, \theta_1 \in [0, L] \).

In order to construct a **counterexample**, in the case \( K > 0 \) choose \( \tilde{K} > K \) such that

\[
\cos \left( L \frac{1}{2} \sqrt{\frac{\tilde{K}}{N}} \right) > \cos \left( L \frac{1}{2} \sqrt{\frac{K}{N - 1}} \right)^{1/N}
\]

Note that such a \( \tilde{K} \) exists since

\[
\cos \left( L \frac{1}{2} \sqrt{\frac{K}{N}} \right) > \cos \left( L \frac{1}{2} \sqrt{\frac{K}{N - 1}} \right)^{1/N}
\]

which in turn is equivalent to \( \sigma_{K,N}^{(1/2)}(L) < \tau_{K,N}^{(1/2)}(L) \), the latter being a general fact, derived in [5], Lemma 1.2. In the case \( K < 0 \), the same argument allows to choose \( \tilde{K} \in (K, 0) \) such that

\[
\cosh \left( L \frac{1}{2} \sqrt{-\frac{K}{N}} \right) > \cosh \left( L \frac{1}{2} \sqrt{-\frac{K}{N - 1}} \right)^{1/N}
\]

Let \( f : [0, L] \to \mathbb{R} \) be any positive solution to the ODE \( f'' = -\frac{\tilde{K}}{N} \cdot f \). Then

\[
f((1 - t)\theta_0 + t\theta_1) = \sigma_{K,N}^{(1-t)}(|\theta_1 - \theta_0|) \cdot f(\theta_0) + \sigma_{K,N}^{(t)}(|\theta_1 - \theta_0|) \cdot f(\theta_1)
\]

for all \( t \in (0, 1) \) and all \( \theta_0, \theta_1 \in [0, L] \). Hence, according to Lemma 1.1 for \( |\theta_0 - \theta_1| \) being sufficiently small

\[
f((1 - t)\theta_0 + t\theta_1) \geq \tau_{K,N}^{(1-t)}(|\theta_1 - \theta_0|) \cdot f(\theta_0) + \tau_{K,N}^{(t)}(|\theta_1 - \theta_0|) \cdot f(\theta_1)
\]

If the Conjecture were true it would then for instance imply

\[
f(L/2) \geq \tau_{K,N}^{(1/2)}(L) \cdot (f(0) + f(L)) = \frac{1}{\cosh \left( L \frac{1}{2} \sqrt{-\frac{K}{N - 1}} \right)^{1/N}} \cdot \frac{f(0) + f(L)}{2},
\]

which...
with appropriate interpretation of the denominator of the RHS in the case $K < 0$. Now let us make a specific choice for $f$, namely, $f(\theta) = \cos \left( (\theta - \frac{L}{2}) \sqrt{\frac{K}{N}} \right)$. Then the previous inequality reads as follows

$$1 \geq \frac{\cos \left( \frac{L}{2} \sqrt{\frac{K}{N}} \right)}{\cos \left( \frac{L}{2} \sqrt{\frac{K}{N-1}} \right)^{1-1/N}}$$

which is a contradiction. \hfill \square

**Remark.** Let us emphasize that the above counterexample is not a counterexample to the local-to-global property of the curvature-dimension condition $CD(K, N)$. It merely says that the way proposed in [6], Theorem 30.37, to prove this local-to-global property will not work.

In the nontrivial case $K/N \neq 0$, it is still an open problem whether the local version of the curvature-dimension condition $CD(K, N)$ implies the corresponding global version.

As one of the main results in [1], the local-to-global property for the reduced curvature-dimension condition $CD^*(K, N)$ was proven for all pairs of $K$ and $N$. Moreover, it was shown that the local versions of $CD(K, N)$ and $CD^*(K, N)$ are equivalent. Hence, the remaining challenge is either prove or disprove that $CD_{loc}(K, N)$ implies $CD(K, N)$ or equivalently

either prove or disprove that $CD^*(K, N)$ implies $CD(K, N)$.

4 A Remark concerning $P_\infty(M, d, m)$ being a Geodesic Space

In Theorem 5.1 of the afore mentioned paper [1], we had assumed that $P_\infty(M, d, m)$ is a geodesic space. This assumption can equivalently be replaced by the much simpler assumption that $\text{supp}[m]$ is a geodesic space. The latter always follows from the preceding (cf. Remark 4.18(ii) in [4]). The converse implication holds true under the assumption of $CD^*_{loc}(K, N)$ for some finite $N$.

Indeed, this implies $CD_{loc}(K -, N)$ with "CD" being defined in the sense of [5]. Due to the non-branching assumption this is equivalent to an analogous "CD" definition in the sense of [2] (Theorem 30.32 in [6] and/or Proposition 4.2 in [5]). The latter in turn implies that $P_\infty(M, d, m)$ is a geodesic space provided $M$ is geodesic with full support (Theorem 30.19(ii) in [6], cf. also proof of Theorem 30.37) or at least if $\text{supp}[m]$ is a geodesic space.

In Theorem 7.10 of [1] the assumption that $m$ has full support has to be added. Then $\hat{M}$ is a geodesic space with full support and the result of Theorem 5.1 applies.

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References


