Extended Version

On the Geometry of Metric Measure Spaces. II.

Karl-Theodor Sturm

This is a continuation of our previous paper [St04][1] 'On the Geometry of Metric Measure Spaces' where we introduced and analyzed lower ('Ricci') curvature bounds $\underline{\mathbb{C}\mathrm{urv}} \geq K$ for metric measure spaces (M, d, m). The definition used there is based on convexity properties of the relative entropy $\mathrm{Ent}(.|m)$ regarded as a function on $\mathcal{P}_2(M, \mathsf{d})$, the L_2-Wasserstein space of probability measures on the metric space (M, d). For Riemannian manifolds, $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ if and only if $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ for all $\xi \in TM$.
This notion of lower curvature bound is a dimension independent (or, in a certain sense, 'infinite dimensional') concept. In order to obtain more precise estimates one has to reinforce the curvature bound $\underline{\mathbb{C}\mathrm{urv}} \geq K$ to a curvature-dimension condition $\mathsf{CD}(K, N)$ involving two parameters K and N playing in some generalized sense the roles of a lower bound for the Ricci curvature and an upper bound for the dimension, resp.

The main topic of the present paper is the *curvature-dimension condition* $\mathsf{CD}(K, N)$ for metric-measure spaces (M, d, m) . In some sense, it will be the geometric counterpart to the curvature-dimension condition for Markov operators and Dirichlet forms by Bakry and Émery [BE85].
We will also study a weak variant of the latter, namely the *measure contraction property* $\mathsf{MCP}(K, N)$. It is a slight modification of a property introduced in [St98] and in a similar form in [KS03].
As in [St04], our definition of the curvature-dimension condition is based on a kind of convexity property for suitable functionals on the L_2-Wasserstein space $\mathcal{P}_2(M, \mathsf{d})$. The previous curvature condition $\underline{\mathbb{C}\mathrm{urv}} \geq K$ is included as the limit case $\mathsf{CD}(K, \infty)$. For finite N the basic object now is the Rényi entropy functional

$$S_N(\rho m|m) = -\int \rho^{1-1/N} dm,$$

replacing the relative ('Shannon') entropy

$$\mathrm{Ent}(\rho m|m) = \int \rho \log \rho \, dm = \lim_{N \to \infty} N(1 + S_N(\rho m|m)).$$

For Riemannian manifolds, the curvature-dimension condition $\mathsf{CD}(K, N)$ will be satisfied if and only if $\dim(M) \leq N$ and $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ for all $\xi \in TM$.
Under minimal regularity assumptions, condition $\mathsf{CD}(K, N)$ will imply property $\mathsf{MCP}(K, N)$; the latter will be strictly weaker. Roughly spoken, $\mathsf{CD}(K, N)$ is a condition on the optimal transport between any pair of (absolutely continuous) probability measures on M whereas $\mathsf{MCP}(K, N)$ is a condition on the optimal transport between Dirac masses and the uniform distribution on M.

One of the fundamental results is that the curvature-dimension condition as well as the measure contraction property are stable under convergence w.r.t. the distance $\mathbb{D}$. The latter was introduced in [St04] as a complete separable metric on the family of (isomorphism classes of)

[1]See also [St05a] for a comprehensive version.

normalized metric measure spaces. Moreover, we deduce that for each triple $(K, N, L) \in \mathbb{R}^3$, the family of normalized metric measure spaces (M, d, m) which have diameter $\leq L$ and which satisfy condition $\mathsf{CD}(K, N)$ (or alternatively, property $\mathsf{MCP}(K, N)$) is compact.

Furthermore, we present various geometric consequences of the curvature-dimension condition (or alternatively of the measure contraction property). The most prominent among them being the Bishop-Gromov theorem on the volume growth of concentric balls and the Bonnet-Myers theorem on the diameter of metric measure spaces with positive lower curvature bounds. In both cases, we obtain the sharp estimates known from the Riemannian case.

Of particular interest are the analytic consequences of property $\mathsf{MCP}(K, N)$. It allows to construct a canonical Dirichlet form and a canonical Laplace operator on $L_2(M, m)$, it implies a local Poincaré inequality, a scale invariant Harnack inequality, and Gaussian estimates for heat kernel and it yields Hölder continuity of harmonic functions.

The curvature-dimension condition can be interpreted as a control on the distortion of infinitesimal volume elements under transport along geodesics. Let us briefly try to explain this.

The curvature-dimension condition $\mathsf{CD}(0, N)$ — which besides the condition $\mathsf{CD}(K, \infty)$ is the easiest to formulate — simply states that for all $N' \geq N$ the functional $S_{N'}(.|m)$ is convex on the L_2-Wasserstein space $\mathcal{P}_2(M, \mathsf{d})$. In the Riemannian case, it was already observed in [St05] that the latter characterizes manifolds with dimension $\leq N$ and Ricci curvature ≥ 0. This is basically due to the fact that the Jacobian determinant $J_t = \det dF_t$ of any 'transport' map $F_t := \exp(-t\nabla\varphi) : M \to M$ satisfies

$$\frac{\partial^2}{\partial t^2} J_t^{1/N}(x) \leq 0 \tag{0.1}$$

if and only if M has dimension $\leq N$ and Ricci curvature ≥ 0. Essentially equivalent to (0.1) is the Brunn-Minkowski inequality:

$$m(A_t)^{1/N'} \geq (1-t) \cdot m(A_0)^{1/N'} + t \cdot m(A_1)^{1/N'} \tag{0.2}$$

for any $N' \geq N$, any $t \in [0, 1]$, and any pair of sets $A_0, A_1 \subset M$ where A_t denotes the set of point γ_t on geodesics with endpoints $\gamma_0 \in A_0, \gamma_1 \in A_1$.

The curvature-dimension condition $\mathsf{CD}(K, N)$ for general K and N is more involved. As a first step, the inequality (0.1) can be replaced by

$$\frac{\partial^2}{\partial t^2} J_t^{1/N}(x) \leq -\frac{K}{N} J_t^{1/N}(x) \cdot \mathsf{d}^2(x, F_1(x))$$

(see [St05], Corollary 3.4). A more refined analysis yields

$$J_t^{1/N}(x) \quad \geq \quad \tau_{K,N}^{(1-t)}(\mathsf{d}(x, F_1(x))) \cdot J_0^{1/N}(x) + \tau_{K,N}^{(t)}(\mathsf{d}(x, F_1(x))) \cdot J_1^{1/N}(x) \tag{0.3}$$

for $t \in [0, 1]$ and $x \in M$ where

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N} \cdot \left(\frac{\sin(\kappa\, t\theta)}{\sin(\kappa\, \theta)}\right)^{1-1/N}$$

with $\kappa = \sqrt{\frac{K}{N-1}}$ (and with appropriate interpretation if $K \leq 0$). The curvature-dimension condition $\mathsf{CD}(K, N)$ to be discussed in the sequel can be regarded as a robust version of (0.3).

Assume for simplicity that for $m \otimes m$-a.e. $(x, y) \in M^2$ there exists a unique geodesic $t \mapsto \gamma_t(x, y)$ depending in a measurable way on the endpoints x and y. Then $\mathsf{CD}(K, N)$ states that that for any pair of absolutely continuous probability measures $\rho_0 m$ and $\rho_1 m$ on M there exists an optimal coupling q such that

$$\rho_t(\gamma_t(x,y)) \le \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y))\,\rho_0^{-1/N}(x) + \tau_{K,N}^{(t)}(\mathsf{d}(x,y))\,\rho_1^{-1/N}(y)\right]^{-N} \tag{0.4}$$

for all $t \in [0,1]$, and q-a.e. $(x,y) \in M^2$ where ρ_t is the density of the push forward of q under the map $(x,y) \mapsto \gamma_t(x,y)$. Roughly spoken, $\mathsf{MCP}(K,N)$ is the particular case where $\rho_0 m$ is degenerated to a Dirac mass. This amounts to say that for every $x \in M$ and $t \in [0,1]$ the 'contracted measure' $m_{t,x} := \gamma_t(x,.)_* \, m$ satisfies

$$t \cdot \left[\frac{\sin(\kappa\;\mathsf{d}(x,y))}{\sin(\kappa\;\mathsf{d}(x,y)/t)}\right]^{N-1} m_{t,x}(dy) \le m(dy). \tag{0.5}$$

In the general case, the assumption of a measurable choice of a unique geodesic is replaced by the assumption of a measurable choice of a measure on the geodesics.

Independently of [St05] and [St04], the particular cases $\mathsf{CD}(K,\infty)$ and $\mathsf{CD}(0,N)$ of the curvature-dimension condition — defined in essentially the same form — were also discussed in a recent paper [LV04] by John Lott and Cédric Villani.

In the first chapter of this paper, we introduce the curvature-dimension condition and we deduce some of the basic properties.
In the second chapter, we derive various geometric consequences of the curvature-dimension condition like Brunn-Minkowski inequality, Bishop-Gromov volume growth estimate, and Bonnet-Myers theorem.
The topic of chapter 3 is the stability of the curvature-dimension condition under convergence. Moreover, compactness of families of normalized metric measure spaces with suitable bounds on the diameter, the dimension and the curvature is deduced.
In chapter 4 we study the curvature-dimension condition under the additional assumption that the underlying space is nonbranching.
In chapter 5 we introduce and study the measure contraction property, including its geometric consequences, and its stability under convergence.
Chapter 6 is devoted to the analytic consequences of the measure contraction property, in particular, the construction of Sobolev spaces and Dirichlet forms as well as the derivation of a scale invariant local Poincaré inequality.

Throughout this paper, we freely use definitions and results from our previous paper [St04]. Theorem x.y and equation (a.b) of that paper will be quoted as Theorem I.x.y or (I.a.b), resp.

1 The Curvature-Dimension Condition

A *metric measure space* will always be a triple (M,d,m) where (M,d) is a complete separable metric space and m is a locally finite measure (i.e. $m(B_r(x)) < \infty$ for all $x \in M$ and all sufficiently small $r > 0$) on M equipped with its Borel σ-algebra. To avoid pathologies, we exclude the case $m(M) = 0$.
$\mathcal{G}(M)$ will denote the space of geodesics $\gamma : [0,1] \to M$, equipped with the topology of uniform convergence. Here and in the sequel by definition each geodesic is minimizing and parametrized proportional to arclength. A point z will be called t-intermediate point of points x and y if $\mathsf{d}(x,z) = t \cdot \mathsf{d}(x,y)$ and $\mathsf{d}(z,y) = (1-t) \cdot \mathsf{d}(x,y)$.

$\mathcal{P}_2(M, \mathsf{d})$ denotes the L_2-Wasserstein space of probability measures on M and d_W the corresponding L_2-Wasserstein distance. The subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(M, \mathsf{d}, m)$.
$\mathbb{X}$ denotes the family of all isomorphism classes of metric measure spaces and $\mathbb{X}_1$ the subfamily of isomorphism classes of normalized metric measure spaces (M, d, m) with finite variances (i.e. $m(M) = 1$ and $\int_M \mathsf{d}^2(o, y)\, dm(y) < \infty$). On $\mathbb{X}_1$ we have introduced the distance $\mathbb{D}$, see chapter 3 in [St04].

Given a metric measure space (M, d, m) and a number $N \in \mathbb{R}$, $N \geq 1$ we define the *Rényi entropy functional*

$$S_N(.|m) : \ \mathcal{P}_2(M, \mathsf{d}) \to \mathbb{R}$$

with respect to m by

$$S_N(\nu|m) := -\int \rho^{-1/N} d\nu$$

where ρ denotes the density of the absolutely continuous part ν^c in the Lebesgue decomposition $\nu = \nu^c + \nu^s = \rho m + \nu^s$ of $\nu \in \mathcal{P}_2(M, \mathsf{d})$. Note that in the borderline case $N = 1$ this reads $S_1(\nu|m) := -m(\mathrm{supp}[\nu^c])$. Instead of S_N, mostly in the literature the functional $\tilde{S}_N := N + N\, S_N$ is considered.The latter shares various properties with the relative Shannon entropy $\mathrm{Ent}(.|m)$. For instance, if m is a probability measure then $\tilde{S}_N(.|m) \geq 0$ on $\mathcal{P}_2(M, \mathsf{d})$ and $\tilde{S}_N(\nu|m) = 0$ if and only if $\nu = m$. For the purpose of this paper, the functional S_N from above is more convenient. We recall two important facts from the proof of Lemma I.4.1.

Lemma 1.1. Assume that $m(M)$ is finite.

(i) Then for each $N > 1$ the Rényi entropy functional $S_N(.|m)$ is lower semicontinuous and satisfies $-m(M)^{1/N} \leq S_N(.|m) \leq 0$ on $\mathcal{P}_2(M, \mathsf{d})$.

(ii) For each $\nu \in \mathcal{P}_2(M, \mathsf{d})$

$$\mathrm{Ent}(\nu|m) = \lim_{N\to\infty} N(1 + S_N(\nu|m)).$$

Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we put for $(t, \theta) \in [0, 1] \times \mathbb{R}_+$:

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq (N-1)\pi^2 \\ t^{1/N}\left(\sin\left(\sqrt{\frac{K}{N-1}}\, t\theta\right) \Big/ \sin\left(\sqrt{\frac{K}{N-1}}\, \theta\right)\right)^{1-1/N}, & \text{if } 0 < K\theta^2 < (N-1)\pi^2 \\ t, & \text{if } K\theta^2 = 0 \text{ or} \\ & \text{if } K\theta^2 < 0 \text{ and } N = 1 \\ t^{1/N}\left(\sinh\left(\sqrt{\frac{-K}{N-1}}\, t\theta\right) \Big/ \sinh\left(\sqrt{\frac{-K}{N-1}}\, \theta\right)\right)^{1-1/N}, & \text{if } K\theta^2 < 0 \text{ and } N > 1. \end{cases}$$

That is, $\tau_{K,N}^{(t)}(\theta) := t^{1/N} \cdot \sigma_{K,N-1}^{(t)}(\theta)^{1-1/N}$ where $\sigma_{K,N}^{(t)}(\theta) := \sin\left(\sqrt{\frac{K}{N}}\, t\theta\right) \Big/ \sin\left(\sqrt{\frac{K}{N}}\, \theta\right)$ if $0 < K\theta^2 < N\pi^2$ and with appropriate interpretation otherwise. Moreover, we put

$$\varsigma_{K,N}^{(t)}(\theta) := \tau_{K,N}^{(t)}(\theta)^N.$$

Straightforward calculations yield that for fixed $t \in]0, 1[$ and $\theta \in]0, \infty[$ the function $(K, N) \mapsto \tau_{K,N}^{(t)}(\theta)$ is continuous, nondecreasing in K and nonincreasing in N. Moreover,

Lemma 1.2. *For all $K, K' \in \mathbb{R}$, all $N, N' \in]0,\infty[$, all $t \in [0,1]$ and all $\theta \in \mathbb{R}_+$:*

$$\sigma_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \sigma_{K+K',N+N'}^{(t)}(\theta)^{N+N'}$$

and, if $N \geq 1$,

$$\tau_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \tau_{K+K',N+N'}^{(t)}(\theta)^{N+N'}.$$

Proof. We derive the first inequality; the rest follows easily. For each fixed $t \in]0,1[$ the function $f : K \mapsto \log \frac{\sin(\sqrt{K}t)}{\sin(\sqrt{K})}$ (with canonical interpretation for nonpositive K) is convex on $]-\infty, \pi^2[$. Hence, for all $K, K' \in \mathbb{R}$, all $N, N' > 0$ and all $\theta \in \mathbb{R}$ under consideration

$$\frac{N}{N+N'} \cdot f\left(\frac{K}{N}\theta^2\right) + \frac{N'}{N+N'} \cdot f\left(\frac{K'}{N'}\theta^2\right) \geq f\left(\frac{K+K'}{N+N'}\theta^2\right).$$

In other words,

$$\left(\frac{\sin\left(\sqrt{\frac{K}{N}}t\theta\right)}{\sin\left(\sqrt{\frac{K}{N}}\theta\right)}\right)^N \cdot \left(\frac{\sin\left(\sqrt{\frac{K'}{N'}}t\theta\right)}{\sin\left(\sqrt{\frac{K'}{N'}}\theta\right)}\right)^{N'} \geq \left(\frac{\sin\left(\sqrt{\frac{K+K'}{N+N'}}t\theta\right)}{\sin\left(\sqrt{\frac{K+K'}{N+N'}}\theta\right)}\right)^{N+N'}.$$

□

In particular, $\tau_{K,N}^{(t)}(\theta) \geq \sigma_{K,N}^{(t)}(\theta)$ provided $N > 1$ since $\tau_{K,N}^N = \sigma_{0,1}^1 \cdot \sigma_{K,N-1}^{N-1}$.

Definition 1.3. Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we say that a metric measure space (M, d, m) satisfies the *curvature-dimension condition* $\mathsf{CD}(K,N)$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exist an optimal coupling q of ν_0, ν_1 and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 with

$$\begin{aligned} S_{N'}(\Gamma(t)|m) \quad \leq \quad -\int_{M\times M} \Big[&\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_0^{-1/N'}(x_0) \\ &+ \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_1^{-1/N'}(x_1)\Big]\, dq(x_0,x_1) \end{aligned} \tag{1.1}$$

for all $t \in [0,1]$ and all $N' \geq N$. Here ρ_i denotes the density of the absolutely continuous part of ν_i w.r.t. m (for $i = 0, 1$).

The definition of the curvature-dimension condition immediately implies its invariance under standard transformations of metric measure spaces. (Cf. also Propositions I.4.12, I.4.13, and I.4.15 as well as the proofs of these results.)

Proposition 1.4. *Let (M, d, m) be a metric measure spaces which satisfies the $\mathsf{CD}(K,N)$ condition for some pair of real numbers K, N. Then the following properties hold:*

(i) 'Isomorphism': *Each metric measure space (M', d', m') which is isomorphic to (M, d, m) satisfies the $\mathsf{CD}(K,N)$ condition.*

(ii) 'Scaled spaces': *For each $\alpha, \beta > 0$ the metric measure space $(M, \alpha\,\mathsf{d}, \beta m)$ satisfies the $\mathsf{CD}(\alpha^{-2}K, N)$ condition.*

(iii) 'Subsets': *For each convex subset M' of M the metric measure space (M', d, m) satisfies the same $\mathsf{CD}(K,N)$ condition.*

Remark 1.5. In order that the curvature-dimension condition is invariant under isomorphisms we require (1.1) to hold true only for $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ and *not for all* $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d})$. For instance, consider $M = \mathbb{R}^n$ equipped with the Euclidean distance d and let m be the $(n-1)$-dimensional Lebesgue measure on $M_0 := \{0\} \times \mathbb{R}^{n-1}$. Then (M, d, m) satisfies the condition $\mathsf{CD}(0, n-1)$. However, choose $\nu_0 = 1_A\, m$ for some set $A \subset M_0$ with $m(A) = 1$ and $\nu_1 = \delta_z$ for some point $z = (z_1, \ldots, z_n)$ with $z_1 \neq 0$. Then for each midpoint $\Gamma_{1/2}$ of them

$$0 = S_{n-1}(\Gamma_{1/2}|m) \not\leq \frac{1}{2} S_{n-1}(\nu_0|m) + \frac{1}{2} S_{n-1}(\nu_1|m) = -\frac{1}{2}.$$

Proposition 1.6.

(i) *If (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ then it also satisfies the curvature-dimension conditions $\mathsf{CD}(K', N')$ for all $K' \leq K$ and $N' \geq N$.*

Conversely, if (M, d, m) is compact with diameter $\leq L$ and satisfies the curvature-dimension conditions $\mathsf{CD}(K_n, N_n)$ for a sequence of pairs (K_n, N_n) with $\lim_{n\to\infty}(K_n, N_n) = (K, N)$ and $K \cdot L^2 < (N-1)\pi^2$ then it also satisfies the curvature-dimension conditions $\mathsf{CD}(K, N)$.

(ii) *If (M, d, m) has finite mass and satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ for some K and N then it has curvature $\geq K$ in the sense of Definition I.4.5.*

Therefore, the condition $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ may be interpreted as the curvature-dimension condition $\mathsf{CD}(K, \infty)$ for (M, d, m).

(iii) *A metric measure space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(0, N)$ for some $N \geq 1$ if and only if the Rényi entropy functionals $S_{N'}(.|m)$ for $N' \geq N$ are weakly convex on $\mathcal{P}_2(M, \mathsf{d}, m)$ in the following sense: for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exist a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 with*

$$S_{N'}(\Gamma(t)|m) \leq (1-t) \cdot S_{N'}(\nu_0|m) + t \cdot S_{N'}(\nu_1|m) \tag{1.2}$$

for all $t \in [0,1]$ and $N' \geq N$.

Proof. (i): The first assertion is obvious. The second one will follow from Theorem 3.1 below.
(ii): Let $\nu_0, \nu_1 \in \mathcal{P}_2(M, d, M)$ be given with $\mathrm{Ent}(\nu_0|m) < \infty$ and $\mathrm{Ent}(\nu_1|m) < \infty$. By assumption, (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ for some K and N. Hence, there exists there exist an optimal coupling q of ν_0, ν_1 and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 with (1.1) for all $t \in [0,1]$ and all $N' \geq N$.
The assumption $m(M) < \infty$ implies that $\mathrm{Ent}(\Gamma_t|m) = \lim_{N'\to\infty} N'(1 + S'_N(\Gamma_t|m))$ for all $t \in [0,1]$. Hence,

$$\begin{aligned}
\mathrm{Ent}(\Gamma_t|m) \quad &- \quad (1-t)\mathrm{Ent}(\Gamma_0|m) - t\mathrm{Ent}(\Gamma_1|m) \\
&= \lim_{N'\to\infty} N' \left(S_{N'}(\Gamma_t|m) - (1-t) S_{N'}(\Gamma_0|m) - t S_{N'}(\Gamma_1|m) \right) \\
&\leq \lim_{N'\to\infty} \int \Big(N' \left[(1-t) - \tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) \right] \rho_0^{-1/N'}(x_0) \\
&\qquad\qquad + N' \left[t - \tau_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1)) \right] \rho_1^{-1/N'}(x_1) \Big)\, dq(x_0, x_1) \\
&\leq \lim_{N'\to\infty} \int \Big(N' \left[(1-t) - \tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) \right] \\
&\qquad\qquad + N' \left[t - \tau_{K,N'}^{(t)}(\mathsf{d}(x_0, x_1)) \right] \Big)\, dq(x_0, x_1) \\
&= -\frac{t(1-t)}{2} K \int \mathsf{d}^2(x_0, x_1)\, dq(x_0, x_1) \;=\; -\frac{t(1-t)}{2} K\, \mathsf{d}_W^2(\nu_0, \nu_1).
\end{aligned}$$

(iii): Obvious. □

Proposition 1.6(iii) gives an elementary characterization of the condition $\mathsf{CD}(0,N)$ through convexity of the Rényi entropy functionals $S_{N'}$ for $N' \geq N$. A generalization of this characterization to $K \neq 0$ will be discussed in chapter 4. Moreover, we will present various modifications of the curvature-dimension condition which formally are more restrictive.
Of course, the most important case to be studied is the case of Riemannian manifolds. Let us mention here the basic result. We postpone its proof to the end of the chapter.

Theorem 1.7. *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and let numbers $K, N \in \mathbb{R}$ with $N \geq 1$ be given.*
(i) *The metric measure space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ if and only if the Riemannian manifold M has Ricci curvature $\geq K$ and dimension $\leq N$.*
(ii) *Moreover, in this case for every measurable function $V : M \to \mathbb{R}$ the weighted space $(M, \mathsf{d}, V\,m)$ satisfies the curvature-dimension condition $\mathsf{CD}(K+K', N+N')$ provided*

$$\mathrm{Hess}\, V^{1/N'} \leq -\frac{K'}{N'} \cdot V^{1/N'}$$

for some numbers $K' \in \mathbb{R}$, $N' > 0$ in the sense that

$$V(\gamma_t)^{1/N'} \geq \sigma^{(1-t)}_{K',N'}(\mathsf{d}(\gamma_0,\gamma_1))\, V(\gamma_0)^{1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(\gamma_0,\gamma_1))\, V(\gamma_1)^{1/N'} \tag{1.3}$$

for each geodesic $\gamma : [0,1] \to M$ and each $t \in [0,1]$.

Let us have a closer look on these results if M is a subset of the real line equipped with the usual distance d and the 1-dimensional Lebesgue measure m.

Example 1.8. **(i)** *For each pair of real numbers $K > 0, N > 1$ the space $([0,L], \mathsf{d}, V\,m)$ with $L := \sqrt{\frac{N-1}{K}}\pi$ and*

$$V(x) = \sin\left(\sqrt{\frac{K}{N-1}}\, x\right)^{N-1}$$

satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$.

(ii) *For each pair of real numbers $K \leq 0, N > 1$ the space $(\mathbb{R}_+, \mathsf{d}, V\,m)$ with*

$$V(x) = \sinh\left(\sqrt{\frac{-K}{N-1}}\, x\right)^{N-1},$$

if $K < 0$, and $V(x) = x^{N-1}$, if $K = 0$, satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$.

(iii) *For each pair of real numbers $K < 0, N > 1$ the space $(\mathbb{R}, \mathsf{d}, V\,m)$ with*

$$V(x) = \cosh\left(\sqrt{\frac{-K}{N-1}}\, x\right)^{N-1}$$

satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$.

Note that for $N \to \infty$ the weight V from example (iii) from above converges to the weight

$$V(x) = \exp\left(\frac{-K}{2}\, x^2\right)$$

from example I.4.10.(ii). Also note that according to [BQ00], the examples (i)-(iii) equipped with natural weighted Laplacians are also the prototypes for the Bakry-Emery curvature-dimension condition.

Proof of Theorem 1.7.
(a) Let M be a complete Riemannian manifold with Ricci curvature $\geq K$ and dimension $n \leq N$ and assume that we are given two absolutely continuous probability measures $\nu_0 = \rho_0 m$ and $\nu_1 = \rho_1 m$ in $\mathcal{P}_2(M, \mathsf{d}, m)$. Without restriction, we may assume that both are compactly supported. (Otherwise, we have to choose compact exhaustions of $M \times M$ and to consider the restriction of the coupling to these compact sets). According to Remark I.2.12(iii), there exists a weakly differentiable function $\varphi : M \to \mathbb{R}$ such that the push forward measures

$$\Gamma_t = (F_t)_* \nu_0$$

with

$$F_t(x) = \exp_x(-t\nabla\varphi(x))$$

for $t \in [0,1]1$ define the unique geodesic $t \mapsto \Gamma_t$ in $\mathcal{P}_2(M, \mathsf{d})$ connecting ν_0 and ν_1. Again each Γ_t is compactly supported and absolutely continuous, say $\Gamma_t = \rho_t m$.
Following [CMS01] we may choose φ in such a way that it is $\mathsf{d}^2/2$-concave[2] and such that for ν_0-a.e. $x \in M$ the Hessian of φ at x exists and the Jacobian $dF_t(x)$ is nonsingular for all $t \in [0,1]$.
(b) For each x and t as above, consider the matrix of Jacobi fields

$$\mathcal{A}_t(x) := dF_t(x) : \; T_x M \to T_{F_t(x)} M$$

along the geodesic $F_\centerdot(x)$. (More precisely, $\mathcal{A}_t(x)v$ is a Jacobi field along $F_\centerdot(x)$ for each $v \in T_x M$.) It is the unique solution of the Jacobi equation

$$\nabla_t \nabla_t \mathcal{A}_t(x) + R\left(\mathcal{A}_t(x), \dot{F}_t(x)\right) \dot{F}_t(x) = 0$$

with initial conditions $\mathcal{A}_0 = Id$, $\nabla_t \mathcal{A}_t|_{\{t=0\}} = -\mathrm{Hess}\,\varphi$. Here R is the curvature tensor and ∇_t denotes covariant derivates along the geodesics $F_\centerdot(x)$ (cf. [Ch93] (3.4)). By assumption, the matrix $\mathcal{A}_t(x)$ is non-degenerate for all x, t under consideration. Hence, the Jacobi equation immediately implies that the self-adjoint matrix valued map $\mathcal{U}_t := \nabla_t \mathcal{A}_t \circ \mathcal{A}_t^{-1}$ solves the Riccati type equation

$$\nabla_t \mathcal{U}_t + \mathcal{U}_t^2 + R\left(\centerdot, \dot{F}_t\right) \dot{F}_t = 0 \tag{1.4}$$

and thus

$$\mathrm{tr}(\nabla_t \mathcal{U}_t) + \mathrm{tr}(\mathcal{U}_t^2) + \mathrm{Ric}\left(\dot{F}_t, \dot{F}_t\right) = 0. \tag{1.5}$$

Now consider $y_t := \log J_t = \log\det \mathcal{A}_t$. Then $\mathrm{tr}\mathcal{U}_t = \mathrm{tr}(\nabla_t \mathcal{A}_t \circ \mathcal{A}_t^{-1}) = \frac{d}{dt}(\log\det \mathcal{A}_t) = \dot{y}_t$ (cf. [Ch93], Prop. 2.8). Hence, $\mathrm{tr}(\nabla_t \mathcal{U}_t) = \frac{d}{dt}\mathrm{tr}(\mathcal{U}_t) = \ddot{y}_t$. By means of the standard estimate $\mathrm{tr}(\mathcal{U}_t^2) \geq \frac{1}{n}(\mathrm{tr}\mathcal{U}_t)^2$ for the trace of the square of a self-adjoint matrix, we obtain from (1.5)

$$\ddot{y}_t \leq -\frac{1}{n}\dot{y}_t^2 - \mathrm{Ric}\left(\dot{F}_t, \dot{F}_t\right). \tag{1.6}$$

Using our estimates for the dimension and the Ricci curvature of M we get $\ddot{y}_t \leq -\frac{1}{N}\dot{y}_t^2 - K\theta^2$ with $\theta(x) := |\dot{F}_t(x)| = \mathsf{d}(x, F_1(x))$ or equivalently

$$\frac{d^2}{dt^2} J_t^{1/N} \leq -\frac{K\theta^2}{N} \cdot J_t^{1/N}. \tag{1.7}$$

[2]This notion of concavity is defined in terms of some generalized Legendre transform. It is completely different from the notion of 'convexity/concavity along geodesics' used at all other instances in this paper.

Integrating (1.7) for fixed x along the geodesic $t \mapsto F_t(x)$ yields

$$J_t^{1/N} \geq \sigma_{K,N}^{(1-t)}(\theta) \cdot J_0^{1/N} + \sigma_{K,N}^{(t)}(\theta) \cdot J_1^{1/N}. \tag{1.8}$$

This is close to the estimate (1.12) which we aim for. In the case $n = 1$ we are already done.

(c) In order to improve upon (1.8) in the case $n \geq 2$, we will separately study the deformation of the volume element in directions parallel and orthogonal to the transport direction. To be precise, fix x as above and let $e_t^1, \ldots, e_t^n$ be an orthonormal basis of $T_{F_t(x)}M$ with $e_t^1 = \dot{F}_t(x) \Big/ |\dot{F}_t(x)|$ for all $t \in [0,1[$. Put $u_{ij}(t) = \langle e_t^i, \mathcal{U}_t e_t^j \rangle$, $\lambda_t = 1 + \int_0^t u_{11}(s)ds$ and $L_t = \exp(\lambda_t)$. Then (1.4) implies

$$-\frac{d}{dt} u_{11}(t) = \sum_{j=1}^n u_{1j}^2(t) \geq u_{11}^2(t). \tag{1.9}$$

That is, $-\ddot{\lambda}_t \geq \dot{\lambda}_t^2$ or, equivalently, $\ddot{L}_t \leq 0$ which in integrated form reads

$$L_t \geq (1-t)L_0 + tL_1 \tag{1.10}$$

for all $t \in [0,1]$. Now put $\alpha_t = y_t - \lambda_t$, $A_t = \exp(\alpha_t) = J_t / L_t$ and $\mathcal{V}_t = (u_{ij}(t))_{i,j=2,\ldots,n}$. Then (1.4) together with (1.9) imply

$$\begin{aligned}
-\ddot{\alpha}_t - K \cdot \theta^2 &\geq -\ddot{y}_t - \mathrm{Ric}\left(\dot{F}_t, \dot{F}_t\right) + \ddot{\lambda}_t \\
&= \mathrm{tr}(\mathcal{U}_t^2) + \ddot{\lambda}_t = \sum_{i,j=1}^n u_{ij}^2(t) - \sum_{j=1}^n u_{1j}^2(t) \\
&\geq \sum_{i,j=2}^n u_{ij}^2(t) = \mathrm{tr}(\mathcal{V}_t^2) \\
&\geq \frac{1}{n-1}(\mathrm{tr}\mathcal{V}_t)^2 = \frac{1}{n-1}\dot{\alpha}_t^2 \geq \frac{1}{N-1}\dot{\alpha}_t^2.
\end{aligned}$$

Hence,

$$\frac{d^2}{dt^2}\left(A_t^{1/(N-1)}\right) \leq -\frac{K\theta^2}{N-1} A_t^{1/(N-1)}$$

and thus

$$A_t^{1/(N-1)} \geq \sigma_{K,N-1}((1-t),\theta) \cdot A_0^{1/(N-1)} + \sigma_{K,N-1}(t,\theta) \cdot A_1^{1/(N-1)}. \tag{1.11}$$

Finally, (1.11) and (1.10) together with Hölder's inequality yield

$$\begin{aligned}
J_t^{1/N} &= (L_t \cdot A_t)^{1/N} \\
&\geq ((1-t)L_0 + tL_1)^{1/N} \cdot \left(\sigma_{K,N-1}^{(1-t)}(\theta) \cdot A_0^{1/(N-1)} + \sigma_{K,N-1}^{(t)}(\theta) \cdot A_1^{1/(N-1)}\right)^{(N-1)/N} \\
&\geq ((1-t)L_0)^{1/N} \cdot \left(\sigma_{K,N-1}^{(1-t)}(\theta) \cdot A_0^{1/(N-1)}\right)^{(N-1)/N} \\
&\quad + (tL_1)^{1/N} \cdot \left(\sigma_{K,N-1}^{(t)}(\theta) \cdot A_1^{1/(N-1)}\right)^{(N-1)/N} \\
&= \tau_{K,N}^{(1-t)}(\theta) \cdot J_0^{1/N} + \tau_{K,N}^{(t)}(\theta) \cdot J_t^{1/N}.
\end{aligned}$$

That is, the Jacobian determinant $J_t(x) := \det dF_t(x)$ satisfies

$$J_t(x)^{1/N} \geq \tau_{K,N}^{(1-t)}(\mathsf{d}(x, F_1(x)) \cdot J_0(x)^{1/N} + \tau_{K,N}^{(t)}(\mathsf{d}(x, F_1(x)) \cdot J_1(x)^{1/N} \tag{1.12}$$

This estimate is – from the technical point of view – the main result in [CMS01] (Lemma 6.1 and Cor. 2.2. To be precise, it is stated there only for the case $N = n$. However, the extension to the general case $N \geq n$ is straightforward.) For the convenience of the reader, we have presented here an alternative, essential self-contained derivation, following similar calculations in [St05].

(d) The change of variable formula for F_t yields that $\rho_t(F_t) \cdot J_t = \rho_0$ a.e. Thus together with (1.12) and (1.3) we obtain

$$\begin{aligned}
S_{N+N'}(\Gamma_t|Vm) &= -\int \left(\frac{\rho_t}{V}\right)^{-\frac{1}{N+N'}} \rho_t dm = -\int J_t^{\frac{1}{N+N'}} V(F_t)^{\frac{1}{N+N'}} \rho_0^{1-\frac{1}{N+N'}} dm \\
&\leq -\int \left(\tau_{K,N}^{(1-t)}(\mathsf{d}) \cdot J_0^{1/N} + \tau_{K,N}^{(t)}(\mathsf{d}) \cdot J_1^{1/N}\right)^{\frac{N}{N+N'}} \\
&\qquad \cdot \left(\sigma_{K',N'}^{(1-t)}(\mathsf{d}) \cdot V(F_0)^{1/N'} + \sigma_{K',N'}^{(t)}(\mathsf{d}) \cdot V(F_1)^{1/N'}\right)^{\frac{N'}{N+N'}} \rho_0^{1-\frac{1}{N+N'}} dm \\
&\leq -\int \Big(\tau_{K,N}^{(1-t)}(\mathsf{d})^{\frac{N}{N+N'}} \cdot \sigma_{K',N'}^{(1-t)}(\mathsf{d})^{\frac{N'}{N+N'}} \cdot J_0^{\frac{1}{N+N'}} \cdot V(F_0)^{\frac{1}{N+N'}} \\
&\qquad + \tau_{K,N}^{(t)}(\mathsf{d})^{\frac{N}{N+N'}} \cdot \sigma_{K',N'}^{(t)}(\mathsf{d})^{\frac{N'}{N+N'}} \cdot J_1^{\frac{1}{N+N'}} \cdot V(F_1)^{\frac{1}{N+N'}}\Big) \rho_0^{1-\frac{1}{N+N'}} dm \\
&\overset{(*)}{\leq} -\int \Big((\tau_{K+K',N+N'}^{(1-t)}(\mathsf{d}) \cdot (J_0\, V(F_0))^{\frac{1}{N+N'}} \\
&\qquad + \tau_{K+K',N+N'}^{(t)}(\mathsf{d}) \cdot (J_1\, V(F_1))^{\frac{1}{N+N'}}\Big) \rho_0^{1-\frac{1}{N+N'}} dm \\
&= -\int \Big((\tau_{K+K',N+N'}^{(1-t)}(\mathsf{d}) \cdot \left(\frac{\rho_0}{V}\right)^{-\frac{1}{N+N'}} \\
&\qquad + \tau_{K+K',N+N'}^{(t)}(\mathsf{d}) \cdot \left(\frac{\rho_1}{V}\right)^{-\frac{1}{N+N'}}\Big) dq
\end{aligned}$$

which proves the claim. Here $(*)$ is due to Lemma 1.2. For the final equality we have used the fact that the unique optimal coupling of ν_0 and ν_1 is given by $dq(x_0, x_1) = \delta_{F_1(x_0)}(dx_1)d\nu(x_0)$. To simplify the above formulae, we always have dropped the arguments x_0 and x_1. To be more specific, d will denote $\mathsf{d}(x_0, F_1(x_0))$ if we integrate with respect to $dm(x_0)$ and it will denote $\mathsf{d}(x_0, x_1)$ if we integrate with respect to $dq(x_0, x_1)$.

(e) Necessity: Assume that (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ for some pair of real numbers K, N with $N \geq 1$. Then according to Corollary 2.5, $N \geq$ Hausdorff dimension of M. In the Riemannian case, the latter coincides with the dimension.

In order to prove that $K \leq$ Ricci curvature of M, let us first investigate the case $n \geq 2$. Assume the contrary: $\mathrm{Ric}_z(\xi, \xi) \leq (K - \delta) \cdot |\xi|^2$ for some $\delta > 0$, some point $z \in M$ and some $\xi \in T_z(M)$. Consider the sets $A_0 = B_\epsilon(\exp_z(-r\xi))$ and $A_1 = B_\epsilon(\exp_z(+r\xi))$. Then for sufficiently small $\epsilon \ll r \ll 1$, Riemannian calculation yields

$$m(A_{1/2}) \leq (1 + \frac{K - \delta}{2} r^2 + O(r^4)) \frac{m(A_0) + m(A_1)}{2}$$

(cf. [St05], Thm. 5.2) whereas Proposition 2.1 gives

$$m(A_{1/2}) \geq (1 + \frac{K}{2} r^2 + O(r^4)) \frac{m(A_0) + m(A_1)}{2}.$$

Hence, $\mathrm{Ric} \geq K$.

Now let us investigate the case $n = 1$. Here the Ricci curvature always vanishes. On the other hand, $n = 1$ implies $N = 1$ (according to Corollary 2.5) and thus necessarily $K \leq 0$ (according to Corollary 2.6, the generalized Bonnet-Myers theorem). Hence, also in this case $\mathrm{Ric} \geq K$. $\square$

2 Geometric Consequences of the Curvature-Dimension Condition

Compared with our defining property (1.1), the following version of the Brunn-Minkowski inequality will be a very weak statement. However, it will still be strong enough to imply all the geometric consequences which we formulate in the sequel.

Proposition 2.1 ('Generalized Brunn-Minkowski Inequality'). *Assume that the metric measure space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ for some real numbers $K, N \in \mathbb{R}$, $N \geq 1$. Then for all measurable sets $A_0, A_1 \subset M$ with $m(A_0) \cdot m(A_1) > 0$, all $t \in [0,1]$ and all $N' \geq N$*

$$m(A_t)^{1/N'} \geq \tau_{K,N'}^{(1-t)}(\Theta) \cdot m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta) \cdot m(A_1)^{1/N'} \tag{2.1}$$

where A_t denotes the set of points which divide geodesics starting in A_0 and ending in A_1 with ratio t:(1-t) and where Θ denotes the minimal/maximal length of such geodesics, that is,

$$A_t := \{y \in M : \exists (x_0, x_1) \in A_0 \times A_1 : \ \mathsf{d}(y, x_0) = t \cdot \mathsf{d}(x_0, x_1), \ \mathsf{d}(y, x_1) = (1-t) \cdot \mathsf{d}(x_0, x_1)\}$$

and

$$\Theta := \begin{cases} \inf_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & \text{if } K \geq 0 \\ \sup_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & \text{if } K < 0. \end{cases}$$

In particular, if $K \geq 0$ then

$$m(A_t)^{1/N'} \geq (1-t) \cdot m(A_0)^{1/N'} + t \cdot m(A_1)^{1/N'}. \tag{2.2}$$

Proof. Let us first assume that $0 < m(A_0) \cdot m(A_1) < \infty$. Applying the curvature-dimension condition $\mathsf{CD}(K,N)$ to $\nu_i := \frac{1}{m(A_i)} 1_{A_i} m$ for $i = 0, 1$ yields

$$\int_{A_t} \rho_t(y)^{1-1/N'} dm(y) \geq \tau_{K,N'}^{(1-t)}(\Theta) \cdot m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta) \cdot m(A_1)^{1/N'} \tag{2.3}$$

where ρ_t denotes the density of some geodesic Γ_t connecting ν_0 and ν_1. (Here without restriction we assume $N' \neq 1$). Now by Jensen's inequality the LHS of (2.3) is dominated by $m(A_t)^{1/N'}$. This proves the claim, provided $m(A_0) \cdot m(A_1) < \infty$. The general case follows by approximation of A_i by sets of finite volume. $\square$

Remark 2.2. The assumption $m(A_0) \cdot m(A_1) > 0$ can not be dropped. For instance, let $M = \mathbb{R}^2$ and let m be the 1-dimensional Lebesgue measure on $A_0 := \{0\} \times \mathbb{R}$ and choose $A_1 = \{1\} \times \mathbb{R}$.

Now let us fix a point $x_0 \in \text{supp}[m]$ and study the growth of the volume of concentric balls

$$v(r) := m(\overline{B}_r(x_0))$$

as well as the growth of the volume of the corresponding spheres

$$s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} \cdot m\left(\overline{B}_{r+\delta}(x_0) \setminus B_r(x_0)\right).$$

Theorem 2.3 ('Generalized Bishop-Gromov Volume Growth Inequality'). *Assume that the metric measure space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ for some real numbers $K, N \in \mathbb{R}$, $N \geq 1$. Then each bounded set $M' \subset M$ has finite volume. Moreover, either m is supported by one point or all points and all spheres have mass 0.*

More precisely, if $N > 1$ then for each fixed $x_0 \in \mathrm{supp}[m]$ and all $0 < r < R \leq \sqrt{\frac{N-1}{K \vee 0}} \cdot \pi$

$$\frac{s(r)}{s(R)} \geq \left(\frac{\sin\left(\sqrt{\frac{K}{N-1}} r\right)}{\sin\left(\sqrt{\frac{K}{N-1}} R\right)} \right)^{N-1} \tag{2.4}$$

and

$$\frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin\left(\sqrt{\frac{K}{N-1}} t\right)^{N-1} dt}{\int_0^R \sin\left(\sqrt{\frac{K}{N-1}} t\right)^{N-1} dt} \tag{2.5}$$

with $s(.)$ and $v(.)$ defined as above and with the usual interpretation of the RHS if $K \leq 0$. In particular, if $K = 0$

$$\frac{s(r)}{s(R)} \geq \left(\frac{r}{R}\right)^{N-1} \quad \text{and} \quad \frac{v(r)}{v(R)} \geq \left(\frac{r}{R}\right)^{N}.$$

The latter also holds true if $N = 1$ and $K \leq 0$.

For each K and each *integer* $N > 1$ the simply connected spaces of dimension N and constant curvature $K/(N-1)$ provide examples where these volume growth estimates are sharp. But also for arbitrary *real* numbers $N > 1$ these estimates are sharp as demonstrated by Example 1.8(i) and (ii) where equality is attained.

Proof. Let us fix a point $x_0 \in \mathrm{supp}[m]$ and assume first that $m(\{x_0\}) = 0$. Let numbers r, R with $0 < r < R$ be given and put $t = \frac{r}{R}$. Choose numbers $\epsilon > 0$ and $\delta > 0$. We will apply the generalized Brunn-Minkowski inequality from above to $A_0 := B_\epsilon(x_0)$ and $A_1 := \overline{B}_{R+\delta R}(x_0) \setminus B_R(x_0)$. One easily verifies that

$$A_t \subset \overline{B}_{r+\delta r+\epsilon r/R}(x_0) \setminus B_{r-\epsilon r/R}(x_0)$$

and $R - \epsilon \leq \Theta \leq R + \delta R + \epsilon$. Hence, Proposition 2.1 implies

$$\begin{aligned} m\left(\overline{B}_{r+\delta r+\epsilon r/R}(x_0) \setminus B_{r-\epsilon r/R}(x_0)\right)^{1/N} \quad &\geq \quad \tau_{K,N}^{(1-r/R)}(R \mp \delta R \mp \epsilon) \cdot m\left(B_\epsilon(x_0)\right)^{1/N} \\ &\quad + \tau_{K,N}^{(r/R)}(R \mp \delta R \mp \epsilon) \cdot m\left(\overline{B}_{R+\delta R}(x_0) \setminus B_R(x_0)\right)^{1/N} \end{aligned}$$

where $\mp$ has to be chosen to coincide with the sign of K. In the limit $\epsilon \to 0$ this yields

$$m\left(\overline{B}_{(1+\delta)r}(x_0) \setminus B_r(x_0)\right)^{1/N} \geq \tau_{K,N}^{(r/R)}((1 \mp \delta)R) \cdot m\left(\overline{B}_{(1+\delta)R}(x_0) \setminus B_R(x_0)\right)^{1/N}$$

or, in other words,

$$v((1+\delta)r) - v(r) \geq \tau_{K,N}^{(r/R)}((1 \mp \delta)R)^N \cdot \left[v((1+\delta)R) - v(R)\right]. \tag{2.6}$$

For r small enough, the LHS will be finite since by assumption m is locally finite and thus $v(R)$ will be finite for all $R \in \mathbb{R}_+$ and it will coincide with $v(R^*)$ for all $R \geq R^* := \sqrt{\frac{N-1}{K \vee 0}} \cdot \pi$. Moreover, by construction v will be right continuous and nondecreasing with at most countably many discontinuities. In particular, there will be arbitrarily small $r > 0$ and $\delta > 0$ such that v is continuous on the interval $[r, (1+\delta)r)$. Hence, by (2.6) v will be continuous on $\mathbb{R}_+$. Therefore, $m(\partial B_r(x_0)) = 0$ for all $r > 0$ and in turn $m(\{x\}) = 0$ for all $x \neq x_0$.

Inequality (2.6) can be restated as

$$\frac{1}{\delta r}\cdot[v((1+\delta)r)-v(r)]\geq\frac{1}{\delta R}\cdot[v((1+\delta)R)-v(R)]\cdot\left(\frac{\sin(\sqrt{\frac{K}{N-1}}(1\mp\delta)r)}{\sin(\sqrt{\frac{K}{N-1}}(1\mp\delta)R)}\right)^{N-1}\tag{2.7}$$

with the usual interpretation if $K\leq 0$. In the limit $\delta\to 0$ this yields the first claim (2.4). Furthermore, given r and δ by successive subdivision of the interval $[r,(1+\delta)r]$ one can construct a sequence $(r_n)_n$ of points in $[r,(1+\delta)r]$ with

$$0\leq\frac{1}{2^{-n}\delta r}\left[v((1+2^{-n}\delta)r)-v(r)\right]\leq\frac{1}{\delta r}\left[v((1+\delta)r)-v(r)\right]=:C.$$

Together with (2.7) this implies that v is locally Lipschitz continuous on $\mathbb{R}_+$. Therefore, in particular, it is weakly differentiable a.e. on $\mathbb{R}_+$ and it coincides with the integral of its weak derivative s. We thus may apply Lemma 3.1 from [Ch93] according to which the inequality (2.4) implies the integrated version (2.5).
It only remains to treat the case $m(\{x_0\})>0$. If there were a point $x_1\in\mathrm{supp}[m]\setminus\{x_0\}$ then we could apply the previous arguments (now with x_1 in the place of x_0) and deduce that $m(\{x\})=0$ for all $x\neq x_1$ which would lead to the contradiction $m(\{x_0\})=0$. Hence, $m(\{x_0\})>0$ implies $\mathrm{supp}[m]=\{x_0\}$. All the estimates of the theorem are trivially true in this case. □

Corollary 2.4 ('Doubling'). *For each metric measure space (M,d,m) which satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ for some real numbers $K,N\in\mathbb{R}$, $N\geq 1$, the doubling property holds on each bounded subset $M'\subset\mathrm{supp}[m]$. In particular, each bounded closed subset $M'\subset\mathrm{supp}[m]$ is compact.*
If $K\geq 0$ or $N=1$ the doubling constant is $\leq 2^N$. Otherwise, it can be estimated in terms of K,N and the diameter L of M' as follows

$$C\leq 2^N\cdot\cosh\left(\sqrt{\frac{-K}{N-1}}\,L\right)^{N-1}.$$

Proof. Assume $N>1$ and put $\kappa=\sqrt{\frac{(-K)\vee 0}{N-1}}$. Then (2.5) immediately yields the doubling property

$$\frac{m(B_{2r}(x))}{m(B_r(x))}\leq\frac{2\int_0^r\sinh(\kappa 2t)^{N-1}\,dt}{\int_0^r\sinh(\kappa t)^{N-1}\,dt}\leq 2^N\cdot\cosh(\kappa r)^{N-1}.$$

The doubling property, however, always implies compactness of the support, see e.g. the proof of Theorem I.3.16. □

Corollary 2.5 ('Hausdorff Dimension'). *For each metric measure space (M,d,m) which satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ for some real numbers $K,N\in\mathbb{R}$, $N\geq 1$, the support of m has Hausdorff dimension $\leq N$.*

Proof. We will prove that for each $N'>N$ the N'-dimensional Hausdorff measure of M vanishes. Without restriction we may assume that M is bounded and that it has full support. For each $\epsilon>0$ we can estimate the ϵ-approximate N'-dimensional Hausdorff measure of M as follows:

$$\begin{aligned}\mathcal{H}^{\epsilon}_{N'}(M) &:= c_{N'}\cdot\inf\left\{\sum_{j=1}^{\infty}\left(\frac{1}{2}\mathrm{diam}S_j\right)^{N'}:\ \bigcup_{j=1}^{\infty}S_j=M,\ \mathrm{diam}S_j\leq\epsilon\right\}\\ &\leq c_{N'}\cdot\epsilon^{N'}\cdot\inf\left\{k\in\mathbb{N}:\ \bigcup_{j=1}^{k}B_\epsilon(x_j)=M\right\}.\end{aligned}$$

According to the doubling property as derived in Corollary 2.4, the minimal number k in the last term can be estimated by $k \leq C \cdot \epsilon^N$ (cf. also proof of Theorem I.4.9). Hence,

$$\mathcal{H}_{N'}(M) := \lim_{\epsilon \to 0} \mathcal{H}^{\epsilon}_{N'}(M) = 0$$

for each $N' > N$. $\square$

Corollary 2.6 ('Generalized Bonnet-Myers Theorem'). *For every metric measure space (M, d, m) which satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ for some real numbers $K > 0$ and $N \geq 1$ the support of m is compact and has diameter*

$$L \leq \sqrt{\frac{N-1}{K}}\,\pi.$$

In particular, if $K > 0$ and $N = 1$ then $\mathrm{supp}[m]$ *consists of one point.*

Proof. Let two points $x_0, x_1 \in \mathrm{supp}[m]$ and $\epsilon > 0$ be given with $\mathsf{d}(x_0, x_1) \geq \sqrt{\frac{N-1}{K}}\,\pi + 4\epsilon$ and $0 < m(B_\epsilon(x_i)) < \infty$. Put $A_i = B_\epsilon(x_i)$ for $i = 0, 1$. Then $A_{1/2} \subset B_R(x_0)$ for some finite R. Hence, Proposition 2.1 with $\Theta > \sqrt{\frac{N-1}{K}}\,\pi$ implies $m(A_{1/2}) = \infty$ whereas Theorem 2.3 implies $m(B_R(x_0)) < \infty$. This contradiction shows that $\mathsf{d}(x_0, x_1) \leq \sqrt{\frac{N-1}{K}}\,\pi$ for all $x_0, x_1 \in \mathrm{supp}[m]$. Finite diameter, however, implies compactness of $\mathrm{supp}[m]$ according to Corollary 2.4. $\square$

3 Stability under Convergence

Theorem 3.1. *Let $((M_n, \mathsf{d}_n, m_n))_{n \in \mathbb{N}}$ be a sequence of normalized metric measure spaces where each $n \in \mathbb{N}$ the space (M_n, d_n, m_n) satisfies the curvature-dimension condition $\mathsf{CD}(K_n, N_n)$ and has diameter $\leq L_n$. Assume that for $n \to \infty$*

$$(M_n, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

and $(K_n, N_n, L_n) \to (K, N, L)$ for some triple $(K, N, L) \in \mathbb{R}^2$ satisfying $K \cdot L^2 < (N-1)\pi^2$. Then the space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ and has diameter $\leq L$.

Corollary 3.2. *For each triple $(K, L, N) \in \mathbb{R}^3$ with $K \cdot L^2 < (N-1)\pi^2$ the family $\mathbb{X}_1(K, N, L)$ of isomorphism classes of normalized metric measure spaces which satisfy the curvature-dimension condition $\mathsf{CD}(K, N)$ and which have diameter $\leq L$ is compact w.r.t. $\mathbb{D}$.*

Proof of the Corollary. Let numbers K, N, L as above be given. The volume growth estimate (2.5) implies that each element in $\mathbb{X}_1(K, N, L)$ satisfies the doubling property with some uniform doubling constant $C = C(K, N, L)$. According to Theorem I.3.16, the family of all (M, d, m) with doubling constant $\leq C$ and diameter $\leq L$ is compact. Hence, it suffices to prove that $\mathbb{X}_1(K, N, L)$ is closed under $\mathbb{D}$-convergence. This is the content of the previous theorem. $\square$

Given $t \in [0, 1], K \in \mathbb{R}$ and $N \geq 1$ we introduce for the sequel the abbreviation

$$T^{(t)}_{K,N}(q|m) = -\int \left[\tau^{(1-t)}_{K,N}(\mathsf{d}(x_0, x_1)) \cdot \rho_0(x_0)^{-1/N} + \tau^{(t)}_{K,N}(\mathsf{d}(x_0, x_1)) \cdot \rho_1(x_1)^{-1/N}\right] dq(x_0, x_1)$$

whenever q is coupling of $\nu_0 = \rho_0 m$ and $\nu_1 = \rho_1 m$.

Lemma 3.3. *Let $K, N \in \mathbb{R}$ with $N > 1$. For each sequence $q^{(k)}$ of optimal couplings with the same marginals ν_0 and ν_1 which converge to some coupling $q^{(\infty)}$*

$$\limsup_{k\to\infty} T^{(t)}_{K,N}(q^{(k)}|m) \le T^{(t)}_{K,N}(q|m).$$

Proof. Let $q^{(k)}$, $k \in \mathbb{N}$, and $q^{(\infty)}$ as above. We will prove that

$$\begin{aligned} \liminf\nolimits_{k\to\infty} \quad & \int \tau^{(1-t)}_{K,N}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-1/N}\, dq^{(k)}(x_0,x_1) \\ & \ge \int \tau^{(1-t)}_{K,N}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-1/N}\, dq^{(\infty)}(x_0,x_1). \end{aligned} \tag{3.1}$$

Together with an analogous assertion with ρ_1 in the place of ρ_0 (and t in the place of $1-t$) this will prove the claim.
For $k \in \mathbb{N} \cup \{\infty\}$ and $C \in \mathbb{R}_+ \cup \{\infty\}$ put

$$v^{(k)}_C(x_0) = \int \left[\tau^{(1-t)}_{K,N}(\mathsf{d}(x_0,x_1)) \wedge C\right] Q^{(k)}(x_0,dx_1)$$

where $Q^{(k)}(x_0,dx_1)$ denotes the disintegration of $dq^{(k)}(x_0,x_1)$ w.r.t. $d\nu(x_0)$. Now fix $C \in \mathbb{R}_+$. Since $\mathcal{C}_b(M)$ is dense in $L_1(M,\nu_0)$ and since $0 \le v^{(k)}_C(.) \le C$, for each $\epsilon > 0$ there exists a function $\psi \in \mathcal{C}_b(M)$ such that

$$\int v^{(k)}_C \cdot \left| \left[\rho_0^{-1/N} \wedge C\right] - \psi \right| d\nu_0 \le \epsilon \tag{3.2}$$

for all $k \in \mathbb{N} \cup \{\infty\}$. The weak convergence $q^{(k)} \to q$ on $M \times M$ implies that there exists a $k(\epsilon) \in \mathbb{N}$ such that for each $k \ge k(\epsilon)$:

$$\int v^{(\infty)}_C \psi\, d\nu_0 \le \int v^{(k)}_C \psi\, d\nu_0 + \epsilon. \tag{3.3}$$

Summing up (3.2) and (3.3) we obtain

$$\begin{aligned} \int v^{(\infty)}_C \cdot \left[\rho_0^{-1/N} \wedge C\right] d\nu_0 \ &\le \ \int v^{(\infty)}_C \cdot \psi\, d\nu_0 + \epsilon \ \le \ \int v^{(k)}_C \cdot \psi\, d\nu_0 + 2\epsilon \\ &\le \ \int v^{(k)}_C \cdot \left[\rho_0^{-1/N} \wedge C\right] d\nu_0 + 3\epsilon \ \le \ \int v^{(k)}_\infty \cdot \rho_0^{-1/N}\, d\nu_0 + 3\epsilon. \end{aligned}$$

That is, for each $C \in \mathbb{R}_+$

$$\int v^{(\infty)}_C \cdot \left[\rho_0^{-1/N} \wedge C\right] d\nu_0 \le \liminf_{k\to\infty} \int v^{(k)}_\infty \cdot \rho_0^{-1/N}\, d\nu_0.$$

Finally, as $C \to \infty$ monotone convergence yields

$$\int v^{(\infty)}_\infty \cdot \rho_0^{-1/N}\, d\nu_0 \le \liminf_{k\to\infty} \int v^{(k)}_\infty \cdot \rho_0^{-1/N}\, d\nu_0.$$

This is precisely our claim (3.1). □

Proof of the Theorem. (i) Let $((M_n, \mathsf{d}_n, m_n))_{n\in\mathbb{N}}$ be a sequence of normalized metric measure spaces as above, each of them satisfying a curvature-dimension condition $\mathsf{CD}(K_n, N_n)$ and having diameter $\le L_n$. Moreover, assume that $(M_n, \mathsf{d}_n, m_n) \longrightarrow (M, \mathsf{d}, m)$ as $n \to \infty$. Then obviously also (M, d, m) has diameter $\le L$. Without restriction, we may assume that $N_n > 1$ and that

there exists a triple (K_0, N_0, L_0) with $K_0 \cdot L_0^2 < (N_0 - 1)\pi^2$ and $K_n \le K_0$, $L_n \le L_0$, $N_n \ge N_0$ for all $n \in \mathbb{N}$. In order to verify the curvature-dimension condition $\mathsf{CD}(K, N)$ let two arbitrary measures $\nu_0 = \rho_0 m$ and $\nu_1 = \rho_1 m$ in $\mathcal{P}_2(M, \mathsf{d}, m)$ and a number $\epsilon > 0$ be given.
(ii) Fix an arbitrary optimal coupling $\tilde{q}$ of them and put $E_r := \{(x_0, x_1) \in M^2 : \rho_0(x_0) < r, \rho_1(x_1) < r\}$, $\alpha_r := \tilde{q}(E_r)$ and $\tilde{q}^{(r)}(.) := \frac{1}{\alpha_r}\tilde{q}(. \cap E_r)$ for $r \in \mathbb{R}_+$. The latter has marginals

$$\tilde{\nu}_0^{(r)}(.) := \tilde{q}^{(r)}(. \times M), \qquad \tilde{\nu}_1^{(r)}(.) := \tilde{q}^{(r)}(M \times .)$$

with bounded densities. Moreover, for sufficiently large $r = r(\epsilon)$

$$\mathsf{d}_W(\nu_0, \tilde{\nu}_0^{(r)}) \le \epsilon, \qquad \mathsf{d}_W(\nu_1, \tilde{\nu}_1^{(r)}) \le \epsilon. \tag{3.4}$$

(iii) Since the densities of $\tilde{\nu}_0^{(r)}$ and $\tilde{\nu}_1^{(r)}$ are bounded there exist a number $R \in \mathbb{R}$ such that

$$\sup_{i=0,1} \mathrm{Ent}(\tilde{\nu}_i^{(r)}|m) + \frac{\sup_n |K_n|}{8} \mathsf{d}_W^2(\tilde{\nu}_0^{(r)}, \tilde{\nu}_1^{(r)}) \le R. \tag{3.5}$$

Choose $n = n(\epsilon) \in \mathbb{N}$ and a coupling $\hat{\mathsf{d}}$ of the metrics d and d_n with

$$\frac{1}{2}\hat{\mathsf{d}}_W(m_n, m) \le \mathbb{D}((M_n, \mathsf{d}_n, m_n), (M, \mathsf{d}, m)) \le \min\left\{\exp\left(-\frac{2 + 4L_0^2 R}{\epsilon^2}\right), \frac{\epsilon}{4C}\right\} \tag{3.6}$$

for some constant C to be specified later. Following the proofs of Lemma I.4.19 and Theorem I.4.20, fix a coupling p of m and m_n which is optimal w.r.t. $\hat{\mathsf{d}}$ and let P and P' be disintegrations of p w.r.t. m and m_n, resp. Recall that P' defines a canonical map $P' : \mathcal{P}_2(M, \mathsf{d}, m) \to \mathcal{P}_2(M_n, \mathsf{d}_n, m_n)$. Put

$$\nu_{i,n} := P'(\tilde{\nu}_i^{(r)}) = \rho_{i,n} m_n$$

with $\rho_{i,n}(y) = \int \tilde{\rho}_i^{(r)}(x) P'(y, dx)$ for $i = 0, 1$. Then (3.5) and (3.6) imply, according to Lemma I.4.19,

$$\hat{\mathsf{d}}_W(\tilde{\nu}_0^{(r)}, \nu_{0,n}) \le \epsilon, \qquad \hat{\mathsf{d}}_W(\tilde{\nu}_1^{(r)}, \nu_{1,n}) \le \epsilon. \tag{3.7}$$

(iv) Due to the curvature-dimension condition on (M_n, d_n, m_n) there exist an optimal coupling q_n of $\nu_{0,n}, \nu_{1,n}$ and a geodesic $\Gamma_{t,n}$ connecting them and satisfying

$$S_{N'}(\Gamma_{t,n}|m_n) \le T_{K',N'}^{(t)}(q_n|m_n) \tag{3.8}$$

for all $N' \ge N_n$, $K' \le K_n$ and $t \in [0, 1]$. Put

$$\Gamma_t^\epsilon = P(\Gamma_{t,n})$$

with $n = n(\epsilon)$ as above and $P : \mathcal{P}_2(M_n, \mathsf{d}_n, m_n) \to \mathcal{P}_2(M, \mathsf{d}, m)$ as introduced in Lemma I.4.19. Then essentially with the same arguments as in the proof of Lemma I.4.19 (now Jensen's inequality applied to the convex function $r \mapsto -r^{1-1/N}$)

$$S_{N'}(\Gamma_t^\epsilon|m) \le S_{N'}(\Gamma_{t,n}|m_n) \tag{3.9}$$

for all N' and t under consideration. Moreover, we know that the curvature-dimension condition $\mathsf{CD}(K_n, N_n)$ implies the curvature bound $\underline{\mathbb{C}\mathrm{urv}}(M_n, \mathsf{d}_n, m_n) \ge K_n$ (cf. Proposition 1.6(ii)) which in turn implies

$$\mathrm{Ent}(\Gamma_t^\epsilon|m) \le \mathrm{Ent}(\Gamma_{t,n}|m_n) \le R.$$

This (together with (3.6)) allows to apply again Lemma I.4.19 to deduce finally

$$\hat{\mathsf{d}}_W(\Gamma_t^\epsilon, \Gamma_{t,n}) \le \epsilon. \tag{3.10}$$

(v) For fixed N', K' and t put

$$v_0(y_0) = \int_{M_n} \tau_{K',N'}^{(1-t)}(\mathsf{d}_n(y_0, y_1))\, Q_n(y_0, dy_1)$$

and

$$v_1(y_1) = \int_{M_n} \tau_{K',N'}^{(t)}(\mathsf{d}_n(y_0, y_1))\, Q_n'(y_1, dy_0)$$

where Q_n and Q_n' are disintegrations of q_n w.r.t. $\nu_{0,n}$ and $\nu_{1,n}$, resp. Then

$$\begin{aligned}
-T_{K',N'}^{(t)}(q_n|m_n) &= \sum_{i=0}^{1} \int_{M_n} \rho_{i,n}(y)^{1-1/N'} \cdot v_i(y)\, dm_n(y) \\
&= \sum_{i=0}^{1} \int_{M_n} \left[\int_M \tilde{\rho}_i^{(r)}(x)\, P'(y, dx)\right]^{1-1/N} \cdot v_i(y)\, dm_n(y) \\
&\ge \sum_{i=0}^{1} \int_{M_n} \int_M \tilde{\rho}_i^{(r)}(x)^{1-1/N}\, P'(y, dx) \cdot v_i(y)\, dm_n(y) \\
&= \sum_{i=0}^{1} \int_M \tilde{\rho}_i^{(r)}(x)^{1-1/N} \left[\int_{M_n} v_i(y)\, P(x, dy)\right] dm(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{M_n} v_0(y_0)\, P(x_0, dy_0) &= \int_{M_n}\int_{M_n} \tau_{K',N'}^{(1-t)}(\mathsf{d}_n((y_0, y_1))Q_n(y_0, dy_1)\, P(x_0, dy_0) \\
&\ge \int_{M_n}\int_{M_n}\int_M \left[\tau_{K',N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) - C \cdot (\mathsf{d}_n(y_0, y_1) - \mathsf{d}(x_0, x_1))\right] \\
&\qquad \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}\, P'(y_1, dx_1)\, Q_n(y_0, dy_1)\, P(x_0, dy_0) \\
&\ge \int_{M_n}\int_{M_n}\int_M \left[\tau_{K',N'}^{(1-t)}(\mathsf{d}(x_0, x_1)) - C \cdot \left(\hat{\mathsf{d}}(x_0, y_0) + \hat{\mathsf{d}}(x_1, y_1)\right)\right] \\
&\qquad \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}\, P'(y_1, dx_1)\, Q_n(y_0, dy_1)\, P(x_0, dy_0)
\end{aligned}$$

where C denotes the maximum of $\frac{\partial}{\partial\theta}\tau_{K',N'}^{(s)}(\theta)$ for $s \in [0,1]$, $N' \ge N_0$, $K' \le K_0$ and $\theta \le L_0$. Analogously,

$$\begin{aligned}
\int_{M_n} v_1(y_1)\, P(x_1, dy_1) &\ge \int_{M_n}\int_{M_n}\int_M \left[\tau_{K',N'}^{(t)}(\mathsf{d}(x_0, x_1)) - C \cdot \left(\hat{\mathsf{d}}(x_0, y_0) + \hat{\mathsf{d}}(x_1, y_1)\right)\right] \\
&\qquad \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}\, P'(y_0, dx_0)\, Q_n'(y_1, dy_0)\, P(x_1, dy_1).
\end{aligned}$$

Define a coupling $\overline{q}^r$ (not necessarily optimal) of $\tilde{\nu}_0^{(r)}$ and $\tilde{\nu}_1^{(r)}$ by

$$\begin{aligned}
d\overline{q}^r(x_0, x_1) &= \int_{M_n\times M_n} \frac{\tilde{\rho}_0^{(r)}(x_0)\tilde{\rho}_1^{(r)}(x_1)}{\rho_{0,n}(y_0)\rho_{1,n}(y_1)}\, P'(y_1, dx_1)\, P'(y_0, dx_0)\, dq_n(y_0, y_1) \\
&= \int_{M_n\times M_n} \frac{\tilde{\rho}_0^{(r)}(x_0)\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}\, P'(y_1, dx_1)\, Q_n(y_0, dy_1)\, P(x_0, dy_0)\, m(dx_0).
\end{aligned}$$

and a coupling q^ϵ of ν_0 and ν_1 by

$$q^\epsilon(.) := \alpha_r \overline{q}^r + \tilde{q}(. \cap (M^2 \setminus E_r))$$

for $r = r(\epsilon)$. Then the above estimates yield

$$\begin{aligned} T^{(t)}_{K',N'}(q_n|m_n) &\leq T^{(t)}_{K',N'}(\overline{q}^r|m) + C\int_M \left[\tilde{\rho}_0^{(r)}(x)^{1-1/N'} + \tilde{\rho}_1^{(r)}(x)^{1-1/N'}\right] \cdot \hat{\mathsf{d}}(x,y)\, dp(x,y) \\ &\leq T^{(t)}_{K',N'}(\overline{q}^r|m) + 2C\, \hat{\mathsf{d}}_W(m,m_n) \;\leq\; T^{(t)}_{K',N'}(\overline{q}^r|m) + \epsilon \end{aligned}$$

due to our choice of n. Moreover,

$$\lim_{\epsilon \to 0} \left| T^{(t)}_{K',N'}(q^\epsilon|m) - T^{(t)}_{K',N'}(\overline{q}^{r(\epsilon)}|m) \right| = 0. \tag{3.11}$$

(vi) Summarizing, we have for each $\epsilon > 0$ a probability measure q^ϵ on M^2 and a family of probability measures Γ^ϵ_t, $t \in [0,1]$ on M satisfying

$$S_{N'}(\Gamma^\epsilon_t|m) \leq S_{N'}(\Gamma_{t,n}|m_n) \leq \; T^{(t)}_{K',N'}(q_n|m_n) \leq T^{(t)}_{K',N'}(\overline{q}^{r(\epsilon)}|m) + \epsilon. \tag{3.12}$$

Compactness of M implies that there exists a sequence $(\epsilon(k))_{k\in\mathbb{N}}$ converging to 0 such that the measures $q^{\epsilon(k)}$ converge to some q and for each rational $t \in [0,1]$ the measures $\Gamma^{\epsilon(k)}_t$ converge to some Γ_t. The measure q has marginals ν_0 and ν_1. According to (3.10), (3.7) and (3.4), it is even an optimal coupling of them.
For each $n \in \mathbb{N}$ the family $\Gamma_{t,n}$, $t \in [0,1]$ is a geodesic in $\mathcal{P}_2(M_n, \mathsf{d}_n, m_n)$ connecting $\nu_{0,n}$ and $\nu_{1,n}$. As $n \to \infty$ the latter converge to ν_0 and ν_1, resp. Together with (3.10) this implies

$$\mathsf{d}_W(\Gamma_s, \Gamma_t) \leq |s-t| \cdot \mathsf{d}_W(\nu_0, \nu_1)$$

for all rational $s, t \in [0,1]$. Hence, the family $(\Gamma_t)_t$ extends to a geodesic connecting ν_0 and ν_1. Moreover, (3.12) and (3.11) together with lower semicontinuity of $S_{N'}(.|m)$ (Lemma 1.1) and upper semicontinuity of $T^{(t)}_{K',N'}(.|m)$ (Lemma 3.3) imply

$$S_{N'}(\Gamma_t|m) \leq \liminf_{k\to\infty} S_{N'}(\Gamma^{\epsilon(k)}_t|m) \leq \liminf_{k\to\infty} T^{(t)}_{K',N'}(q^{\epsilon(k)}|m) \leq T^{(t)}_{K',N'}(q|m) \tag{3.13}$$

for all $t \in [0,1]$, all $N' > N = \lim_n N_n$ and all $K' < K = \lim_n K_n$. By continuity of $S_{N'}$ and $T^{(t)}_{K',N'}$ in (K',N') the inequality $S_{N'}(\Gamma_t|m) \leq T^{(t)}_{K',N'}(q|m)$ also holds for $(K',N') = (K,N)$. This proves the Theorem. $\square$

4 Nonbranching Spaces

Several aspects of optimal mass transportation become much simpler if the underlying space is *nonbranching* in the sense of Definition I.2.8. In this chapter, we will study the curvature-dimension condition for nonbranching spaces.

Lemma 4.1. *Assume that (M, d, m) is nonbranching and satisfies condition $\mathsf{CD}(K,N)$ for some pair (K,N). Then for every $x \in \text{supp}[m]$ and m-a.e. $y \in M$ (with exceptional set depending on x) there exists a unique geodesic between x and y.*
Moreover, there exists a measurable map $\gamma : M^2 \to \mathcal{G}(M)$ such that for $m \otimes m$-a.e. $(x,y) \in M^2$ the curve $t \mapsto \gamma_t(x,y)$ is the unique geodesic connecting x and y.

Proof. Fix $x_0 \in M$, $t \in]0,1[$ and some closed set $A_1 \subset M$. Let A_t^n for $n \in \mathbb{N}$ denote the set of all t-intermediate points $z = \gamma_t(x,y)$ between points $x \in B_{1/n}(x_0)$ and $y \in A_1$. Assume without restriction $m(B_{1/n}(x_0)) \cdot m(A_1) > 0$. According to Corollary 2.1

$$m(A_t^n) \geq \inf_{x \in B_{1/n}(x_0), y \in A_1} \varsigma_{K,N}^{(t)}(\mathsf{d}(x,y)) \cdot m(A_1)$$

for each n and thus (as $n \to \infty$)

$$m(A_t) \geq \inf_{y \in A_1} \varsigma_{K,N}^{(t)}(\mathsf{d}(x,y)) \cdot m(A_1)$$

with $A_t := \bigcap_n A_t^n$.

Each point $z \in A_t$ lies on some geodesic starting in x_0 and ending somewhere in A_1. Indeed, for each n the point z will be a t-intermediate point of some $x_n \in B_{1/n}(x_0)$ and some $y_n \in A_1$. By local compactness of M and closedness of A_1 there exists a point $y_0 \in A_1$ such that (after passing to a suitable subsequence) $y_n \to y_0$ and thus z will also be a t-intermediate point of x_0 and y_0.

Now choose $A_1 = \overline{B}_R(x_0)$ for some large R. Decomposing A_1 into a disjoint union $\bigcup_i A_1^i$ with $A_1^i = A_1 \cap \left(\overline{B}_{\epsilon i}(x_0) \setminus \overline{B}_{\epsilon(i-1)}(x_0)\right)$ and applying the previous estimate to each of the A_1^i yields (as $\epsilon \to 0$)

$$m(A_t) \geq \int_{A_1} \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y))\, m(dy)$$

where A_t denotes the set of t-intermediate points between x_0 and some $y \in A_1$. Nonbranching of M therefore will imply that for each $z \in A_t$ the geodesic from x_0 to z is unique.
Now $\varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y)) \to 1$ as $t \to 1$ for all $y \in M$ with $K \cdot \mathsf{d}^2(x_0,y) < (N-1)\pi^2$ and $\varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y)) = \infty$ and for all other $y \in M$. Hence, $m(A_1 \setminus \bigcup_{t<1} A_t) = 0$ and thus for m-a.e. $z \in A_1$ there exists a unique geodesic connecting x_0 and z. Finally, for $R \to \infty$ this yields the claim concerning uniqueness of geodesics.

For the claim concerning the measurable choice of geodesics (or intermediate points), fix a number $t \in]0,1[$ and assume for simplicity $m(M) = 1$. For each $k \in \mathbb{N}$ let $M = \bigcup_i M_{i,k}$ be a (finite or countable) covering of M by measurable sets $M_{i,k}$ with diameter $\leq 1/k$ and $\lambda_{i,k} := m(M_{i,k}) > 0$. Let $p_{i,k}$ be a probability measure on M^3 with the following properties: the projection $(\pi_1)_* p_{i,k}$ onto the first component is the probability measure $\frac{1}{\lambda_{i,k}} 1_{M_{i,k}} m$; the projection on the third component is m; the joint distribution of the first and third component is an optimal coupling of them; and conditioned under the first and third component, the second component is a t-intermediate point of them.
Hence, for each k the probability measure $p_k := \sum_i \lambda_{i,k} p_{i,k}$ on M^3 has the following properties: the projection on the first component is m; the projection on the third component is m; and conditioned under the first and third component, the second component is a t-intermediate point of them.
Now by compactness there exists an accumulation point p of the $p_k, k \in \mathbb{N}$. It has the following properties: the joint projection on the first and third component is $m \otimes m$; and conditioned under the first and third component, the second component is a t-intermediate point of them. That is, $p(A \times M \times C) = m(A) \times m(C)$ for all measurable $A, C \subset M$; moreover, for p-a.e. $(x,z,y) \in M^3$ the point z is a t-intermediate point of x and y. Disintegration of measures yields a Markov kernel P from M^2 to M such that

$$dp(x,z,y) = P(x,y;dz) m(dx) m(dy).$$

According to the uniqueness of t-intermediate points

$$P(x,y;dz) = \delta_{\gamma_t(x,y)}(dz)$$

for m^2-a.e. (x,y). This finally proves the measurability of γ_t since by definition P is measurable in (x,y). $\square$

Proposition 4.2. *Given numbers $K \in \mathbb{R}$ and $N \geq 1$ and a compact nonbranching metric measure space (M, d, m). Then the following are equivalent:*

(i) *(M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$;*

(ii) *For each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exist a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 and an optimal coupling q such that for all $t \in [0,1]$ and all $N' \geq N$*

$$S_{N'}(\Gamma(t)|m) \leq \tau_{K,N'}^{(1-t)}(\Theta) \cdot S_{N'}(\nu_0|m) + \tau_{K,N'}^{(t)}(\Theta) \cdot S_{N'}(\nu_1|m) \tag{4.1}$$

where $\Theta := \left\{ \begin{array}{ll} q\text{-essinf}_{x_0,x_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0 \\ q\text{-esssup}_{x_0,x_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0 \end{array} \right\}$ denotes the $\left\{ \begin{array}{l} \text{minimal} \\ \text{maximal} \end{array} \right\}$ transportation distance.

(iii) *For each pair of points $z_0, z_1 \in M$ there exists an $\epsilon > 0$ such that for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ with $\mathrm{supp}[\nu_0] \subset B_\epsilon(z_0)$, $\mathrm{supp}[\nu_1] \subset B_\epsilon(z_1)$ there exist an optimal coupling q and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting them and satisfying (1.1).*

(iv) *For each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ and each optimal coupling q of them*

$$\rho_t(\gamma_t(x_0,x_1)) \leq \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x_0,x_1))\, \rho_0^{-1/N}(x_0) + \tau_{K,N}^{(t)}(\mathsf{d}(x_0,x_1))\, \rho_1^{-1/N}(x_1) \right]^{-N} \tag{4.2}$$

for all $t \in [0,1]$, and q-a.e. $(x_0,x_1) \in M^2$. Here ρ_t is the density of the push forward of q under the map $(x_0,x_1) \mapsto \gamma_t(x_0,x_1)$. It is determined by $\int_M u(y)\rho_t(y)\, dm(y) = \int_{M\times M} u(\gamma_t(x_0,x_1))\, dq(x_0,x_1)$ for all bounded measurable $u : M \to \mathbb{R}$.

(v) *For each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ and* each *optimal coupling q of them there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 and satisfying (1.1) for all $t \in [0,1]$ and all $N' \geq N$.*

Proof. (i) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (i), and (v) $\Rightarrow$ (i): trivial.
(i)$\Rightarrow$ (ii): Immediate consequence of the fact that $\tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \geq \tau_{K,N'}^{(t)}(\Theta)$ for all $t \in [0,1]$ and all x_0, x_1 with $K\,\mathsf{d}(x_0,x_1) \geq K\Theta$. (Actually, this implication does not require that M is compact and nonbranching.)

(iii) $\Rightarrow$ (i): By compactness of M, there exist finitely many disjoint sets $L_1, \ldots, L_n$ which cover M such that for each pair $i,j \in \{1,\ldots,n\}$ and each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ with $\mathrm{supp}[\nu_0] \subset \overline{L}_i$, $\mathrm{supp}[\nu_1] \subset \overline{L}_j$ there exist an optimal coupling q and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting them and satisfying (1.1).
Now let arbitrary ν_0 and $\nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ be given. Fix an arbitrary optimal coupling $\tilde{q}$ of them and define probability measures ν_0^{ij} and ν_1^{ij} for $i,j = 1,\ldots,n$ by

$$\nu_0^{ij}(A) := \frac{1}{\alpha_{ij}} \tilde{q}((A \cap L_i) \times L_j) \quad \text{and} \quad \nu_1^{ij}(A) := \frac{1}{\alpha_{ij}} \tilde{q}(L_i \times (A \cap L_j)) \tag{4.3}$$

provided $\alpha_{ij} := \tilde{q}(L_i \times L_j) \neq 0$. Then $\mathrm{supp}[\nu_0^{ij}] \subset \overline{L}_i$ and $\mathrm{supp}[\nu_1^{ij}] \subset \overline{L}_j$. Therefore, for each pair $(i,j) \in \{1,\ldots,n\}^2$ the assumption can be applied to the probability measures ν_0^{ij} and ν_1^{ij}.

It yields the existence of an optimal coupling q^{ij} of them and of a geodesic Γ^{ij} connecting them with the property

$$\begin{aligned} S_{N'}(\Gamma_t^{ij}|m) \quad &\leq \quad - \int_{M\times M} \Big[\tau_{K,N'}^{(1-t)}(\,\mathsf{d}(x_0,x_1))\, \rho_0^{ij}(x_0)^{-1/N'} \\ &\qquad\qquad + \; \tau_{K,N'}^{(t)}(\,\mathsf{d}(x_0,x_1))\, \rho_1^{ij}(x_1)^{-1/N'} \Big]\, dq^{ij}(x_0,x_1) \end{aligned} \tag{4.4}$$

for all $t\in[0,1]$ and all $N'\geq N$. Define

$$q := \sum_{i,j=1}^n \alpha_{ij} q^{ij}, \quad \Gamma_t := \sum_{i,j=1}^n \alpha_{ij}\Gamma_t^{ij}. \tag{4.5}$$

Then q is an optimal coupling of ν_0 and ν_1 and Γ is a geodesic connecting them. Moreover, since the $\nu_0^{ij}\otimes\nu_1^{ij}$ for different choices of $(i,j)\in\{1,\ldots,n\}^2$ are mutually singular and since M is nonbranching, also the Γ_t^{ij} for different choices of $(i,j)\in\{1,\ldots,n\}^2$ are mutually singular, Lemma I.2.11(iii) (for each fixed $t\in[0,1]$). Hence,

$$S_{N'}(\Gamma_t|m) = \sum_{i,j} \alpha_{ij}^{1-1/N'} \cdot S_{N'}(\Gamma_t^{ij}|m)$$

and one simply may sum up both sides of inequality (4.4) – multiplied by $\alpha_{ij}^{1-1/N'}$ – to obtain the claim.

(ii)$\Rightarrow$(i): Let numbers K,N and a compact nonbranching space (M,d,m) with the property (4.1) be given. Moreover, let two measures $\nu_0=\rho_0$ and $\nu_1=\rho_1 m\in\mathcal{P}_2(M,\mathsf{d},m)$ be given and choose an arbitrary optimal coupling $\tilde q$ of them. For each $\epsilon>0$ choose a finite covering $(L_i)_{i=1,\ldots,n}$ of M by sets L_i of diameter $\leq\epsilon/2$. Define numbers α_{ij} and probability measures ν_0^{ij} and ν_1^{ij} for $i,j=1,\ldots,n$ as in the previous proof. Then by assumption there exist an optimal coupling q^{ij} of them and a geodesic Γ^{ij} connecting them with the property

$$\begin{aligned} S_{N'}(\Gamma_t^{ij}|m) \quad &\leq \quad - \int_{M\times M} \Big[\tau_{K,N'}^{(1-t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon)\, \rho_0^{ij}(x_0)^{-1/N'} \\ &\qquad\qquad + \; \tau_{K,N'}^{(t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon)\, \rho_1^{ij}(x_1)^{-1/N'} \Big]\, dq^{ij}(x_0,x_1) \end{aligned} \tag{4.6}$$

for all $t\in[0,1]$ and all $N'\geq N$ and with $\mp$ depending on the sign of K. Then for each $\epsilon>0$ as before $q^{(\epsilon)} := \sum_{i,j=1}^n \alpha_{ij}q^{ij}$ defines an optimal coupling of ν_0 and ν_1 and $\Gamma_t^{(\epsilon)} := \sum_{i,j=1}^n \alpha_{ij}\Gamma_t^{ij}$ defines a geodesic connecting ν_0 and ν_1. Compactness of M implies that there exists a sequence $(\epsilon(k))_{k\in\mathbb{N}}$ converging to 0 such that $q^{(\epsilon(k))}$ converge to some q and such that the geodesics $\Gamma_{\cdot}^{(\epsilon(k))}$ converge to some geodesic $\Gamma_{\cdot}$ in $\mathcal{P}_2(M,\mathsf{d},m)$. Hence, for each fixed $\epsilon'>0$ and all t and $N'>1$ under consideration

$$\begin{aligned} S_{N'}(\Gamma_t|m) \quad &\leq \quad \liminf_{k\to\infty} S_{N'}(\Gamma_t^{(\epsilon(k))}|m) \\ &\leq \quad -\limsup_{k\to\infty} \int_{M\times M} \Big[\tau_{K,N'}^{(1-t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon')\, \rho_0^{(\epsilon(k))}(x_0)^{-1/N'} \\ &\qquad\qquad + \; \tau_{K,N'}^{(t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon')\, \rho_1^{(\epsilon(k))}(x_1)^{-1/N'} \Big]\, dq^{(\epsilon(k))}(x_0,x_1) \\ &\leq \quad - \int_{M\times M} \Big[\tau_{K,N'}^{(1-t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon')\, \rho_0(x_0)^{-1/N'} \\ &\qquad\qquad + \; \tau_{K,N'}^{(t)}(\,\mathsf{d}(x_0,x_1)\mp\epsilon')\, \rho_1(x_1)^{-1/N'} \Big]\, dq(x_0,x_1) \end{aligned}$$

where the last inequality is proven similarly as Lemma 3.3. Finally, by monotone convergence the claim follows as $\epsilon' \to 0$.

(i)⇒(iv): Assume that the $\mathsf{CD}(K,N)$ condition holds and that M is compact and nonbranching. Let measures ν_0 and ν_1 be given as well as an optimal coupling $\tilde{q}$ of them.
Choose a $\cap$-stable generator $\{M_n\}_{n\in\mathbb{N}}$ of the Borel σ-field of M with $m(\partial M_n) = 0$ for all n. For each $n \in \mathbb{N}$ consider the disjoint covering of M by the 2^n sets $L_1 = M_1 \cap \dots M_n$, $L_2 = M_1 \cap \dots M_{n-1} \cap \complement M_n$, $\dots$, $L_{2^n} = \complement M_1 \cap \dots \cap \complement M_n$. For each fixed n, define probability measures ν_0^{ij} and ν_1^{ij} as in (4.3) (proof of the implication '(iii)⇒(i)') and choose optimal couplings q^{ij} of them with

$$\int_{M\times M} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_0^{ij}(x_0)^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_1^{ij}(x_1)^{-1/N'}\right] dq^{ij}(x_0,x_1)$$
$$\leq \int_M \rho_t^{ij}(\gamma_t(x_0,x_1))^{-1/N'}\, dq^{ij}(x_0,x_1). \tag{4.7}$$

Define as in (4.5)

$$q^{(n)} := \sum_{i,j=1}^{2^n} \alpha_{ij} q^{ij}. \tag{4.8}$$

Then by construction for all $i,j \leq n$

$$\int_{M_i\times M_j} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_0^{(n)}(x_0)^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_1^{(n)}(x_1)^{-1/N'}\right] dq^{(n)}(x_0,x_1)$$
$$\leq \int_{M_i\times M_j} \rho_t^{(n)}(\gamma_t(x_0,x_1))^{-1/N'}\, dq^{(n)}(x_0,x_1). \tag{4.9}$$

Compactness of M implies that – at least along a suitable subsequence – the $q^{(n)}$ converge to an optimal coupling q of ν_0 and ν_1. Since $m(\partial M_i) = 0$ for all i, we obtain for all $i,j \in \mathbb{N}$

$$q(M_i \times M_j) = \lim_{n\to\infty} q^{(n)}(M_i \times M_j) = \tilde{q}(M_i \times M_j).$$

Hence, $q = \tilde{q}$. Moreover, we may apply (modifications of) Lemma 1.1 and Lemma 3.3 to pass to the limit in (4.9) and to obtain

$$\int_{M_i\times M_j} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_0(x_0)^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_1(x_1)^{-1/N'}\right] dq(x_0,x_1)$$
$$\leq \int_{M_i\times M_j} \rho_t(\gamma_t(x_0,x_1))^{-1/N'}\, dq(x_0,x_1).$$

Since this holds for all i,j it finally implies

$$\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_0(x_0)^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1)) \cdot \rho_1(x_1)^{-1/N'} \leq \rho_t(\gamma_t(x_0,x_1))^{-1/N'}$$

for q-a.e. $(x_0,x_1) \in M^2$. With the particular choice $N' = N$ this is (iv).

(iv)⇒(v): We will prove that estimate (4.2) for a given N implies the corresponding estimate for any $N' \geq N$. Indeed, by Hölder's inequality and Lemma 1.2

$$\begin{aligned}
\rho_t^{-1/N'}(\gamma_t(x_0,x_1)) &\geq \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x_0,x_1))\,\rho_0^{-1/N}(x_0) + \tau_{K,N}^{(t)}(\mathsf{d}(x_0,x_1))\,\rho_1^{-1/N}(x_1)\right]^{N/N'} \\
&\geq \tau_{K,N}^{(1-t)}(\mathsf{d}(x_0,x_1))^{N/N'} \cdot (1-t)^{1-N/N'} \cdot \rho_0^{-1/N'}(x_0) \\
&\qquad + \tau_{K,N}^{(t)}(\mathsf{d}(x_0,x_1))^{N/N'} \cdot t^{1-N/N'} \cdot \rho_1^{-1/N'}(x_1) \\
&\geq \tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\,\rho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\,\rho_1^{-1/N'}(x_1).
\end{aligned}$$

Finally, integrating this estimate with respect to the given optimal coupling q yields estimate (1.1). $\square$

5 The Measure Contraction Property

Recall from chapter 1 that

$$\varsigma_{K,N}^{(t)}(\theta) = t \cdot \left[\sin\left(\sqrt{\frac{K}{N-1}}\, t\theta\right) \Big/ \sin\left(\sqrt{\frac{K}{N-1}}\, \theta\right) \right]^{N-1}$$

if $0 < K\theta^2 < (N-1)\pi^2$ and with appropriate interpretations otherwise.

Definition 5.1. Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we say that a metric measure space (M, d, m) satisfies *the measure contraction property* $\mathsf{MCP}(K, N)$ iff for each $0 < t < 1$ there exists a Markov kernel P_t from M^2 to M such that for m^2-a.e. (x, y) and for $P_t(x, y; .)$-a.e. z the point z is a t-intermediate point of x and y and such that for m-almost every $x \in M$ and for every measurable $B \subset M$

$$\int_M \varsigma_{K,N}^{(t)}\left(\mathsf{d}(x,y)\right) P_t(x,y;B)\, dm(y) \leq m(B), \tag{5.1}$$

$$\int_M \varsigma_{K,N}^{(1-t)}\left(\mathsf{d}(x,y)\right) P_t(y,x;B)\, dm(y) \leq m(B). \tag{5.2}$$

Lemma 5.2. *A metric measure space (M, d, m) satisfies the measure contraction property $\mathsf{MCP}(K, N)$ if and only if for each $0 < t < 1$ there exists a measure p_t on M^3 such that for p_t-a.e. (x, z, y) the point z is a t-intermediate point of x and y and such that for all measurable sets $A, B, C \subset M$*

$$p_t(A \times M \times C) = m(A) \cdot m(C), \tag{5.3}$$

$$\int_{A\times B\times M} \varsigma_{K,N}^{(t)}\left(\mathsf{d}(x,y)\right) dp_t(x,z,y) \leq m(A) \cdot m(B), \tag{5.4}$$

$$\int_{M\times B\times C} \varsigma_{K,N}^{(1-t)}\left(\mathsf{d}(x,y)\right) dp_t(x,z,y) \leq m(B) \cdot m(C). \tag{5.5}$$

Proof. $(i) \Leftrightarrow (ii)$: Given the Markov kernel P_t define a measure p_t as follows:

$$dp_t(x, z, y) = P_t(x, y; dz)m(dx)m(dy).$$

Vice versa, given the measure p_t define the Markov kernel P_t with the above properties by means of disintegration of measures. $\square$

In the case $K = 0$ the previous conditions simply read as

$$\begin{aligned} p_t(A \times M \times C) &\leq m(A) \cdot m(C), \\ t^N \cdot p_t(A \times B \times M) &\leq m(A) \cdot m(B), \\ (1-t)^N \cdot p_t(M \times B \times C) &\leq m(B) \cdot m(C). \end{aligned}$$

For alternative formulations of conditions (6.1) (and (6.2)) see Remark 6.11 below.

Remark 5.3. Most of the results of chapter 2 also remain true with condition $\mathsf{MCP}(K, N)$ in the place of condition $\mathsf{CD}(K, N)$. In particular, this is the case for

- Theorem 2.3, Generalized Bishop-Gromov Volume Growth Inequality;
- Corollary 2.4, Doubling;
- Corollary 2.5, Hausdorff Dimension;
- Corollary 2.6 Generalized Bonnet-Myers Theorem.

The proofs are essentially the same. Actually, for all these geometric consequences, property (5.2) is not required (i.e. the so-called one-sided MCP suffices, see Remark 5.11).

Under minimal regularity assumptions on (M, d, m) condition $\mathsf{CD}(K,N)$ implies $\mathsf{MCP}(K,N)$. These regularity assumptions are *either* that M is nonbranching *or* that geodesics in M are unique (at least for m^2-a.e. pair of endpoints). Indeed, the latter assumption will follow from the former (Lemma 4.1).

Theorem 5.4. *Assume that there exists a measurable map $\gamma : M^2 \to \mathcal{G}(M)$ such that for $m \otimes m$-a.e. $(x,y) \in (\mathrm{supp}[m])^2$ the curve $\gamma_\bullet(x,y)$ is the unique geodesic connecting x and y. Then condition $\mathsf{CD}(K,N)$ implies property $\mathsf{MCP}(K,N)$.*

Proof. Let $\gamma : M^2 \to \mathcal{G}(M)$ as above and define for each $t \in [0,1]$ a Markov kernel P_t from M^2 to M by

$$P_t(x,y;B) := 1_B(\gamma_t(x,y))$$

and for each t,x a measure $m_{t,x} = \int P_t(x,y;.)m(dy)$ on M by

$$m_{t,x}(B) := \int_M 1_B(\gamma_t(x,y))\, m(dy).$$

For each $x \in M$ let M_x denote the set of all $y \in M$ for which there exists a unique geodesic connecting x and y and let M_0 be the set of x such that $m(M \setminus M_x) = 0$. By assumption $m(M \setminus M_0) = 0$.
Now assume $\mathsf{CD}(K,N)$. Fix a point $x_0 \in M_0$ and a closed subset $B \subset M$. Then by inner regularity of m there exist closed sets $M_k \subset M_{x_0}$ with $m(M \setminus M_k) \le \frac{1}{k}$. Put $A_0^n := B_{1/n}(x_0)$, $A_1^k := \gamma_t(x_0,.)^{-1}(B) \cap M_k$. Moreover, let $A_t^{n,k}$ denote the set of all $\gamma_t(x,y)$ with $x \in A_0^n, y \in A_1^k$. Then for each k

$$\bigcap_n A_t^{n,k} \subset B. \tag{5.6}$$

Indeed, assume that $z \in \bigcap_{n\in\mathbb{N}} A_t^{n,k}$, i.e. there exist $x_n \in B_{1/n}(x_0), y_n \in A_1^k$ with $z = \gamma_t(x_n,y_n)$ for all n. Then by definition of A_1^k there exist $w_n \in B$ with $w_n = \gamma_t(x_0,y_n)$ for all n. Now by local compactness of M (cf. Corollary 2.4) and by closedness of M_k and B there exist $y_0 \in M_k, w_0 \in B$ such that (after passing to suitable subsequences) $y_n \to y_0$ and $w_n \to w_0$ as $n \to \infty$. Moreover, we have $x_n \to x_0$. Hence, z as well as w_0 will be t-intermediate points of x_0 and y_0. By uniqueness of intermediate points, $z = w_0$ and thus $z \in B$.
Apply our version of the Brunn-Minkowski inequality, Proposition 2.1, to the sets A_0^n and A_1^k. It yields

$$m(A_t^{n,k}) \ge \inf_{x\in A_0^n, y\in A_1^k} \varsigma_{K,N}^{(t)}(\mathsf{d}(x,y)) \cdot m(A_1^k)$$

for all $k,n \in \mathbb{N}$. This, together with (5.6), implies as $n \to \infty$

$$m(B) \ge \inf_{y\in A_1^k} \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y)) \cdot m(A_1^k)$$

for all k which in turn implies as $k \to \infty$

$$m(B) \geq \inf_{y \in \gamma_t(x_0,.)^{-1}(B)} \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y)) \cdot m\left(\gamma_t(x_0,.)^{-1}(B)\right) = \inf_{z \in B} \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,z)/t) \cdot m_{t,x_0}(B).$$

Decomposing B into a disjoint union $\bigcup_i B_i$ with $B_i = B \cap \left(\overline{B}_{\epsilon i}(x_0) \setminus \overline{B}_{\epsilon(i-1)}(x_0)\right)$ and applying the previous estimate to each of the B_i finally yields (as $\epsilon \to 0$)

$$m(B) \geq \int_B \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,z)/t)\; m_{t,x_0}(dz)$$

or equivalently

$$m(B) \geq \int_M \varsigma_{K,N}^{(t)}(\mathsf{d}(x_0,y)) \cdot P_t(x_0,y;B)\, m(dy).$$

Finally, by inner regularity these estimates carry over from all closed sets B to all measurable sets $B \subset M$. $\square$

The assumptions of the previous Theorem are in particular satisfied in the Riemannian case. The proof of the 'only if' part of the assertion follows the argumentation in the proof of Theorem 1.7.

Corollary 5.5 ('Riemannian Spaces'). *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and let numbers $K, N \in \mathbb{R}$ with $N \geq 1$ be given.*
(i) *If the Riemannian manifold M has Ricci curvature $\geq K$ and dimension $\leq N$ then the metric measure space (M, d, m) satisfies property $\mathsf{MCP}(K,N)$.*
Moreover, in this case for every measurable function $V : M \to \mathbb{R}$ the weighted space $(M, \mathsf{d}, V\, m)$ satisfies property $\mathsf{MCP}(K+K', N+N')$ provided

$$\mathrm{Hess}\, V^{1/N'} \leq -\frac{K'}{N'} \cdot V^{1/N'}$$

for some numbers $K' \in \mathbb{R}$, $N' > 0$ in the sense of (1.3).
(ii) *Conversely, if (M, d, m) satisfies property $\mathsf{MCP}(K,N)$ then M has dimension $\leq N$.*
If (M, d, m) satisfies property $\mathsf{MCP}(K,n)$ where n denotes the dimension of M, then M has Ricci curvature $\geq K$.

In general, property $\mathsf{MCP}(K,N)$ for a Riemannian manifold will *not imply* that M has Ricci curvature $\geq K$. This can be seen from the following

Remark 5.6. [3] *For each $N > 1$ there exists a constant $c_N > 0$ such that each compact Riemannian manifold M with Ricci curvature ≥ 0, dimension $\leq N-1$ and diameter $\leq L$ satisfies property*

$$\mathsf{MCP}(K,N) \quad \text{for each positive } K \leq c_N/L^2.$$

Proof. According to part (i) of the previous theorem, the space (M, d, m) satisfies property $\mathsf{MCP}(0, N-1)$. It therefore suffices to prove that

$$t^{N-1} \geq \varsigma_{K,N}^{(t)}(\theta)$$

for all $t \in [0,1]$ and $\theta \in [0,L]$. Now for sufficiently small $c_N \in]0,1]$ and all $K\theta^2 \leq c_N$ the right hand side can be estimated from above by $t^N \cdot (1 + (1-t^2)K\theta^2/2)$. But obviously

$$t^{N-1} \geq t^N \cdot (1 + (1-t^2)K\theta^2/2)$$

for all $K\theta^2 \leq 1$. $\square$

[3] As observed by S. Ohta, this remark (applied e.g. to small convex subsets of $\mathbb{R}^{N-1}$) proves that $\mathsf{MCP}(K,N)$ is *strictly* weaker than $\mathsf{CD}(K,N)$. It also proves that $\mathsf{MCP}(K,N)$ is *not* a local property.

Theorem 5.7 ('Alexandrov Spaces'). *Let M be a complete locally compact geodesic space with curvature $\geq \kappa$ in the sense of Alexandrov and with finite Hausdorff dimension n for some numbers $\kappa \in \mathbb{R}, n \in \mathbb{N}$. Let m be the n-dimensional Hausdorff measure on M. Then the metric measure space (M, d, m) satisfies property $\mathsf{MCP}((n-1)\kappa, n)$.*

Proof. Proposition 6.1, Lemma 6.1 and Theorem 6.1 in [KS01]. □

Remark 5.8. Let (M, d, m) be a metric measure space which satisfies property $\mathsf{MCP}(K, N)$ for some pair of real numbers K, N. Then the following properties hold:

(i) 'Isomorphism': Each metric measure space (M', d', m') which is isomorphic to (M, d, m) satisfies property $\mathsf{MCP}(K, N)$.

(ii) 'Scaled spaces': For each $\alpha, \beta > 0$ the metric measure space $(M, \alpha\,\mathsf{d}, \beta m)$ satisfies property $\mathsf{MCP}(\alpha^{-2}K, N)$.

(iii) 'Subsets': For each convex subset M' of M the metric measure space (M', d, m) satisfies property $\mathsf{MCP}(K, N)$.

(iv) 'Hierarchy': (M, d, m) satisfies conditions $\mathsf{MCP}(K', N')$ for all $K' \leq K$ and $N' \geq N$.

Theorem 5.9 ('Stability under Convergence'). *Let $((M_n, \mathsf{d}_n, m_n))_{n\in\mathbb{N}}$ be a sequence of normalized metric measure spaces where for each $n \in \mathbb{N}$ the space (M_n, d_n, m_n) satisfies property $\mathsf{MCP}(K_n, N_n)$ and has diameter $\leq L_n$. Assume that for $n \to \infty$*

$$(M_n, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

and $(K_n, N_n, L_n) \to (K, N, L)$ for some triple $(K, N, L) \in \mathbb{R}^3$. Then the space (M, d, m) satisfies property $\mathsf{MCP}(K, N)$ and has diameter $\leq L$.

Proof. Let a sequence of normalized metric measure spaces $((M_n, \mathsf{d}_n, m_n))_{n\in\mathbb{N}}$, and a limit space (M, d, m) be given as above. Passing to a suitable subsequence we may assume that there exist a metric measure space $(\hat{M}, \hat{\mathsf{d}}, \hat{m})$ and a sequence of probability measures $(\hat{m}_n)_{n\in\mathbb{N}}$ on $\hat{M}$, weakly converging to $\hat{m}$, such that for each $n \in \mathbb{N}$ the space (M_n, d_n, m_n) is isomorphic to the space $(\hat{M}, \hat{\mathsf{d}}, \hat{m}_n)$ and the space (M, d, m) is isomorphic to the space $(\hat{M}, \hat{\mathsf{d}}, \hat{m})$, ([St04], part (v) in the proof of Theorem 3.6, or 'Union Lemma' $3\frac{1}{2}$.12 in [Gr99] together with Lemma 3.7 in [St04]). Without restriction, we therefore may assume $M_n = \hat{M} = M$, $\mathsf{d}_n = \hat{\mathsf{d}} = \mathsf{d}$ for all n as well as $\hat{m}_n = m_n$ for all n and $\hat{m} = m$.
Property $\mathsf{MCP}(K, N)$ for (M, d, m_n) implies the ('restricted') doubling property (Corollary 2.4 and Remark 5.3) and thus compactness of M_n. Since doubling constant and diameter can be estimated uniformly in n, we may also assume that M is compact ([St04], Theorem 3.16 and Lemma 3.15).

Property $\mathsf{MCP}(K, N)$ for (M, d, m_n) states that for each $t \in]0, 1[$ there exists a probability measure $p_t^{(n)}$ on M^3 satisfying (5.3), (5.4), and (5.5). The latter conditions are equivalent to the fact that for all nonnegative bounded measurable functions u on M^2

$$\int_{M^3} u(x,y)\, dp_t^{(n)}(x,z,y) \;=\; \int_{M^2} u(x,y)\, dm_n(x)\, dm_n(y) \tag{5.7}$$

$$\int_{M^3} u(x,z)\, dp_t^{(n)}(x,z,y) \;\leq\; \int_{M^2} \frac{u(x,z)}{\varsigma^{(t)}(\mathsf{d}(x,z)/t)}\, dm_n(x)\, dm_n(z) \tag{5.8}$$

$$\int_{M^3} u(z,y)\, dp_t^{(n)}(x,z,y) \;\leq\; \int_{M^2} \frac{u(z,y)}{\varsigma^{(t)}(\mathsf{d}(z,y)/(1-t))}\, dm_n(z)\, dm_n(y). \tag{5.9}$$

Since $p_t^{(n)}$ and m_n are Radon measures on the compact space M, it suffices to verify (5.7), (5.8), and (5.9) for all nonnegative bounded continuous functions u on M^2. Note that also the function $1/\varsigma^{(t)}(\mathsf{d}(x,y)/t)$ is bounded continuous on M^2.
Compactness of M implies that there exists a probability measure p_t on M^3 such that (after passing to an appropriate subsequence) $p_t^{(n)} \to p_t$ weakly as $n \to \infty$. Together with the weak convergence of $m_n \to m$ it implies that

$$\begin{aligned} \int_{M^3} u(x,y)\, dp_t(x,z,y) &= \int_{M^2} u(x,y)\, dm(x)\, dm(y) \\ \int_{M^3} u(x,z)\, dp_t(x,z,y) &\le \int_{M^2} \frac{u(x,z)}{\varsigma^{(t)}(\mathsf{d}(x,z)/t)}\, dm(x)\, dm(z) \\ \int_{M^3} u(z,y)\, dp_t(x,z,y) &\le \int_{M^2} \frac{u(z,y)}{\varsigma^{(t)}(\mathsf{d}(z,y)/(1-t))}\, dm(z)\, dm(y). \end{aligned}$$

for all nonnegative bounded continuous functions u on M. As mentioned before, this is equivalent to properties (5.3), (5.4), and (5.5) for p_t. $\square$

Corollary 5.10 ('Compactness'). *For each triple $(K,L,N) \in \mathbb{R}^3$ the family $\mathbb{X}_1(K,N,L)$ of isomorphism classes of normalized metric measure spaces which satisfy property $\mathsf{MCP}(K,N)$ and which have diameter $\le L$ is compact w.r.t. $\mathbb{D}$.*

Proof. Analogous to the proof of Corollary 3.2. $\square$

Let us discuss some more conditions related to property $\mathsf{MCP}(K,N)$.

Remark 5.11. Given a metric measure space (M, d, m) the following properties are equivalent:

(i) For each $0 < t < 1$ there exists a Markov kernel P_t from M^2 to M such that for every $x \in \mathrm{supp}[m]$, for m-a.e. y and for $P_t(x,y;.)$-a.e. z the point z is a t-intermediate point of x and y and such that estimate (5.1) holds true for every $x \in \mathrm{supp}[m]$ and every measurable subset $B \subset M$.

(ii) For each $0 < t < 1$ there exists a Markov kernel P_t from M^2 to M such that for m^2-a.e. (x,y) and for $P_t(x,y;.)$-a.e. z the point z is a t-intermediate point of x and y and such that estimate (5.1) holds true for m-a.e. $x \in M$ and every measurable subset $B \subset M$.

(iii) There exists a measure Υ on $\mathcal{G}(M)$ such that $\Upsilon_{0,1} = m \otimes m$ and for each $t \in [0,1]$

$$\varsigma_{K,N}^{(t)}\left(\frac{\mathsf{d}(.,.)}{t}\right)\, \Upsilon_{0,t} \;\le\; m \otimes m. \tag{5.10}$$

Here $\Upsilon_{s,t} = (\pi_{s,t})_* \Upsilon$ for $s,t \in [0,1]$ is the push forward of Υ under the projection $\pi_{s,t} : \mathcal{G}(M) \to M^2, \gamma \mapsto (\gamma_s, \gamma_t)$.

In this case, we say that (M, d, m) satisfies the *one-sided measure contraction property* $\mathsf{MCP}^{1/2}(K,N)$.

Proof. (i)⇒(ii) and (iii)⇒(ii) are trivial. (ii)⇒(i) follows by weak convergence of $\int P_t(x_n,y;.)m(dy) \to \int P_t(x,y;.)m(dy)$ as $x_n \to x \in \mathrm{supp}[m]$.
It remains to prove (ii)⇒(iii). According to Remark 5.3 (and Corollary 2.4) we may assume without restriction that M is compact. For each $n \in \mathbb{N}$ we define a measure $p^{(n)}$ on M^{n+1} by

$$\begin{aligned} &dp^{(n)}(x_0, x_1, \ldots, x_k, \ldots, x_n) \\ &\quad = P_{\frac{1}{2}}(x_0, x_2; dx_1) \ldots P_{\frac{k-1}{k}}(x_0, x_k; dx_{k-1}) \ldots P_{\frac{n-1}{n}}(x_0, x_n; dx_{n-1})\, dm(x_n)\, dm(x_0). \end{aligned}$$

Passing to a suitable subsequence, by compactness there exists a limit Υ, such that for infinitely many $n \in \mathbb{N}$

$$\left(\pi_{0,\frac{1}{n},\ldots,\frac{n-1}{n},1}\right)_\star \Upsilon = p^{(n)}$$

as measures on M^{n+1}. In particular, this implies $(\pi_{0,1})_* \Upsilon = m \otimes m$.
Moreover, for all such n, all $k = 1, \ldots, n-1$ and all measurable sets $A_0, A_k \subset M$ we obtain (using estimate (5.1))

$$\begin{aligned}
&\left(\varsigma_{K,N}^{(\frac{k}{n})} \left(\frac{\mathsf{d}(.,.)}{k/n} \right) \cdot (\pi_{0,k/n})_* \Upsilon \right) (A_0 \times A_k) \\
= \quad & \int \ldots \int \varsigma_{K,N}^{(\frac{k}{n})} \left(\frac{\mathsf{d}(x_0, x_k)}{k/n} \right) \cdot 1_{A_0}(x_0) \cdot 1_{A_k}(x_k) \\
& \qquad P_{\frac{1}{2}}(x_0, x_2; dx_1) \ldots P_{\frac{k-1}{k}}(x_0, x_k; dx_{k-1}) \ldots P_{\frac{n-1}{n}}(x_0, x_n; dx_{n-1})\, dm(x_n)\, dm(x_0) \\
\overset{(5.1)}{\leq} \quad & \int \ldots \int \varsigma_{K,N}^{(\frac{k}{n})} \left(\mathsf{d}(x_0, x_n) \right) \cdot 1_{A_0}(x_0) \cdot 1_{A_k}(x_k) \cdot \varsigma_{K,N}^{(\frac{n-1}{n})} \left(\mathsf{d}(x_0, x_n) \right)^{-1} \\
& \qquad P_{\frac{1}{2}}(x_0, x_2; dx_1) \ldots P_{\frac{n-2}{n-1}}(x_0, x_{n-1}; dx_{n-2})\, dm(x_{n-1})\, dm(x_0) \\
\overset{(5.1)}{\leq} \quad & \ldots \\
\leq \quad & \int \ldots \int 1_{A_0}(x_0) \cdot 1_{A_k}(x_k) \;\; P_{\frac{1}{2}}(x_0, x_2; dx_1) \ldots P_{\frac{k-1}{k}}(x_0, x_k; dx_{k-1})\, dm(x_k)\, dm(x_0) \\
= \quad & m(A_0) \cdot m(A_k).
\end{aligned}$$

This already proves estimate (5.10) for all $t = \frac{k}{n}$ as above. By weak convergence then the claim follows for all $t \in [0, 1]$. □

Remark 5.12. Assume that the metric measure space (M, d, m) satisfies condition $\mathsf{MCP}(K, N)$. Then there exists a measure Υ on $\mathcal{G}(M)$ such that $\Upsilon_{0,1} = m \otimes m$ and $\forall n \in \mathbb{N}$, $\forall k = 1, \ldots, 2^n$

$$\varsigma_{K,N}^{(2^{-n})} \left(2^n \, \mathsf{d}(.,.) \right) \; \Upsilon_{(k-1)2^{-n}, k2^{-n}} \quad \leq \quad m \otimes m. \tag{5.11}$$

Proof. The measure Υ is obtained as the projective limit of measures $p^{(n)}$ on M^{2^n+1} defined recursively by $p^{(0)} = m \otimes m$ and

$$dp^{(n)}(x_0, x_1, \ldots, x_{2^n}) = P_{2^{-n}}(x_0, x_2; dx_1) \ldots P_{2^{-n}}(x_{2^n-2}, x_{2^n}; dx_{2^n-1})\, dp^{(n-1)}(x_0, x_2, \ldots, x_{2^n}).$$

□

6 Analytic Consequences of the Measure Contraction Property

Throughout this chapter, assume that the metric measure space (M, d, m) satisfies property $\mathsf{MCP}(K, N)$ for some $K, N \in \mathbb{R}$.
For $p \in [1, \infty[$ and $u \in L_p(M, m)$ define the 'p-th order energy integral' by

$$\mathcal{E}_N^p(u) := \sup_{\varphi \in \mathcal{C}_c(M), \varphi \leq 1} \mathcal{E}_N^p(u, \varphi) \tag{6.1}$$

where

$$\mathcal{E}_N^p(u, \varphi) := \limsup_{r \to 0} \frac{N}{r^N} \int \int_{B_r^*(x)} \left| \frac{u(x) - u(y)}{\mathsf{d}(x, y)} \right|^p \varphi(x)\, dm(y)\, dm(x). \tag{6.2}$$

Moreover, define the 'p-th order Sobolev space' by

$$W^{1,p}(M) := \{ u \in L_p(M, m) : \; \mathcal{E}_N^p(u) < \infty \}.$$

Theorem 6.1. *(i) For each* $p \in [1, \infty[$*, each* $u \in W^{1,p}(M)$ *and each* $\varphi \in \mathcal{C}_c(M)$ *the limit*

$$\lim_{r \to 0} \frac{N}{r^N} \int \int_{B_r^*(x)} \left| \frac{u(x) - u(y)}{\mathsf{d}(x,y)} \right|^p \varphi(x) \, dm(y) \, dm(x)$$

exists and coincides with $\mathcal{E}_N^p(u, \varphi)$*.*
(ii) For each p *and* u *as above there exists a measure* $\mu_N^p(u,.)$ *on* M *('p-th order energy measure') such that*

$$\mathcal{E}_N^p(u, \varphi) = \int_M \varphi(x) \mu_N^p(u, dx)$$

for all $\varphi \in \mathcal{C}_c(M)$*.*
(iii) For each $p \in [1, \infty[$*, the energy integral* $u \mapsto \mathcal{E}_N^p(u)$ *is lower semicontinuous on* $L_p(M, m)$*, and* $W^{1,p}(M)$ *is a Banach space.*

The proof of the theorem follows the argumentation in [KS93], [St98] and [KS01]. The key ingredient is the following lemma which uses approximation of $\mathcal{E}_N^p$ by functionals slightly different from those in the theorem. Fix a nonnegative measurable function η on $\mathbb{R}$ with $\mathrm{supp}[\eta] \subset [0,1]$ and $\int_{\mathbb{R}} \eta(s) ds = 1$. (The choice $\eta = N \cdot s^{N-1} \cdot 1_{[0,1]}$ covers the situation from the theorem.) Put

$$\mathcal{E}_{K,N}^{p,r}(u, \varphi) := \frac{1}{r} \int \int \left| \frac{u(x) - u(y)}{\mathsf{d}(x,y)} \right|^p \cdot \eta\left(\frac{\mathsf{d}(x,y)}{r} \right) \cdot s_{K,N}(\mathsf{d}(x,y))^{1-N} \cdot \varphi(x) \, dm(y) \, dm(x)$$

where $s_{K,N}(\Theta) := \sqrt{\frac{N-1}{K}} \cdot \sin\left(\sqrt{\frac{K}{N-1}} \Theta \right)$ (with the usual interpretation if $K \leq 0$).

Lemma 6.2 ('Subpartitioning Lemma'). *For all* $p \in [1, \infty[$*,* $u \in W^{1,p}(M)$ *and* $\varphi \in \mathcal{C}_c(M)$*,* $\varphi \geq 0$*, and for all* $0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = 1$

$$\mathcal{E}_{K,N}^{p,r}(u, \varphi) \leq \sum_{i=1}^n (t_i - t_{i-1}) \cdot \mathcal{E}_{K,N}^{p,(t_i - t_{i-1})r}(u, \varphi_r) \tag{6.3}$$

where $\varphi_r(x) = \varphi(x) + \sup_{y \in B_r(x)} |\varphi(x) - \varphi(y)|$*.*

Proof. For simplicity here we restrict ourselves to the case $n = 2$. The general case follows analogously, see [St98].
Let us fix $t \in]0,1[$ and let $P_t(x, y; dz)$ be the Markov kernel from M^2 to M from the definition of the measure contraction property $\mathsf{MCP}(K,N)$. Then

$$\begin{aligned}
\mathcal{E}_{K,N}^{p,r}(u, \varphi) &= \frac{1}{r} \int \int \left| \frac{u(x) - u(y)}{\mathsf{d}(x,y)} \right|^p \eta\left(\frac{\mathsf{d}(x,y)}{r} \right) s_{K,N}(\mathsf{d}(x,y))^{1-N} \varphi(x) \, dm(y) \, dm(x) \\
&\overset{(*)}{\leq} \frac{1}{r} \int \int \int \left[t^{1-p} \left| \frac{u(x) - u(z)}{\mathsf{d}(x,y)} \right|^p + (1-t)^{1-p} \left| \frac{u(z) - u(y)}{\mathsf{d}(x,y)} \right|^p \right] P_t(x, y; dz) \\
&\qquad \cdot \eta\left(\frac{\mathsf{d}(x,y)}{r} \right) s_{K,N}(\mathsf{d}(x,y))^{1-N} \varphi(x) \, dm(y) \, dm(x) \\
&\leq \frac{1}{r} \int \int \int \left| \frac{u(x) - u(z)}{\mathsf{d}(x,z)} \right|^p \eta\left(\frac{\mathsf{d}(x,z)}{tr} \right) t \, s_{K,N}(\mathsf{d}(x,y))^{1-N} P_t(x, y; dz) \, dm(y) \, \varphi_r(x) \, dm(x) \\
&\quad + \frac{1}{r} \int \int \int \left| \frac{u(z) - u(y)}{\mathsf{d}(z,y)} \right|^p \eta\left(\frac{\mathsf{d}(z,y)}{(1-t)r} \right) (1-t) \, s_{K,N}(\mathsf{d}(x,y))^{1-N} P_t(x, y; dz) \, dm(x) \, \varphi_r(y) \, dm(y) \\
&\overset{(**)}{\leq} \frac{1}{r} \int \int \left| \frac{u(x) - u(z)}{\mathsf{d}(x,z)} \right|^p \eta\left(\frac{\mathsf{d}(x,z)}{tr} \right) s_{K,N}(\mathsf{d}(x,z))^{1-N} \, dm(z) \, \varphi_r(x) \, dm(x) \\
&\quad + \frac{1}{r} \int \int \left| \frac{u(z) - u(y)}{\mathsf{d}(z,y)} \right|^p \eta\left(\frac{\mathsf{d}(z,y)}{(1-t)r} \right) s_{K,N}(\mathsf{d}(z,y))^{1-N} \, dm(z) \, \varphi_r(y) \, dm(y) \\
&= t \cdot \mathcal{E}_{K,N}^{p,tr}(u, \varphi_r) + (1-t) \cdot \mathcal{E}_{K,N}^{p,(1-t)r}(u, \varphi_r).
\end{aligned}$$

Here $(*)$ is due to the standard estimate $(a+b)^p \le t^{1-p}a^p + (1-t)^{1-p}b^p$ and $(**)$ is due to our assumption (5.1).

□

Proof of the Theorem. We only sketch the main steps. See [KS93], [St98] and [KS01] for more details.
(i) Using the estimate of the lemma, one verifies that for each p, u and φ as above the limit $\lim_{r\to 0} \mathcal{E}^{p,r}_{K,N}(u,\varphi)$ exists and is independent of the choice of η and of K. It therefore coincides with $\mathcal{E}^p_N(u,\varphi)$.
(ii) For each $p \ge 1$ and each $u \in W^{1,p}(M)$ the map $\varphi \mapsto \mathcal{E}^p_N(u,\varphi)$ is linear. It is represented by a measure $\mu^p_N(u,dx)$ (independent of K and η) which is the weak limit of the measures $\rho^{p,r}_{K,N}(u,x)\,dm(x)$ for $r \to 0$ where

$$\rho^{p,r}_{K,N}(u,x) := \frac{1}{r}\int \left|\frac{u(x)-u(y)}{\mathsf{d}(x,y)}\right|^p \cdot \eta\left(\frac{\mathsf{d}(x,y)}{r}\right) \cdot s_{K,N}(\mathsf{d}(x,y))^{1-N}\,dm(y).$$

(iii) Now choose $\eta(s) = (p+N)\cdot s^{p+N-1}\cdot 1_{[0,1]}(s)$. Then the densities $\rho^{p,r}_{K,N}(u,.)$ of the approximate energy measures are given by

$$\rho^{p,r}_{K,N}(u,x) = \frac{p+N}{r^{p+N}}\int_{B_r(x)} |u(x)-u(y)|^p \cdot \left(\frac{\mathsf{d}(x,y)}{s_{K,N}(\mathsf{d}(x,y))}\right)^{N-1} dm(y).$$

Hence, if $(u_k)_k$ is a sequence in $W^{1,p}(M)$ converging in $L_p(M,m)$ to some $u \in L_p$ then $\mathcal{E}^{p,r}_{K,N}(u_k,\varphi) \to \mathcal{E}^{p,r}_{K,N}(u,\varphi)$ for each $r>0$ and φ as $k\to\infty$. Now assume $\sup_k \mathcal{E}^p_N(u_k) < C$. Then for all $\varphi \le 1$, all sufficiently small $r>0$, and all $k\in\mathbb{N}$

$$\mathcal{E}^{p,r}_{K,N}(u_k,\varphi) < C.$$

Thus $\mathcal{E}^p_N(u) \le C$. Passing to a suitable subsequence finally yields

$$\liminf_{k\to\infty} \mathcal{E}^p_N(u_k) \;\ge\; \mathcal{E}^p_N(u).$$

□

The last statement of the preceding theorem also admits an extension to varying state spaces. Let (M_i, d_i, m_i) for $i\in\mathbb{N}$ be a family of normalized metric measure spaces with

$$(M_i, \mathsf{d}_i, m_i) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

as $i\to\infty$. Given functions $u_i \in L_p(M_i,m_i)$ and $u\in L_p(M,m)$ we say that

$$u_i \to u \quad \text{in } L_p$$

iff there exist a family of couplings q_i of the measures m_i and m and a family of couplings $\hat{\mathsf{d}}_i$ of the metrics d_i and d such that $\int_{M_i\times M} \hat{\mathsf{d}}_i(x,y)^2\,dq_i(x,y) \to 0$ and

$$\int_{M_i\times M} |u_i(x)-u(y)|^p\;dq_i(x,y) \to 0 \qquad \text{as } i\to\infty.$$

Theorem 6.3. *Let (M_i, d_i, m_i), $i\in\mathbb{N}$, be a family of normalized compact metric measure spaces satisfying the measure contraction property* $\mathsf{MCP}(K,N)$ *for some pair (K,N) and converging to* *a metric measure space (M, d, m). For $p \in [1, \infty[$ and $i \in \mathbb{N}$, let $\mathcal{E}_N^{i,p}$ and $\mathcal{E}_N^p$ be the p-th order energy integral on $L_p(M_i, m_i)$ and $L_p(M, m)$, resp. Then*

$$\liminf_{i\to\infty} \mathcal{E}_N^{i,p}(u_i) \geq \mathcal{E}_N^p(u)$$

for all $u \in L_p(M, m)$ and all sequences of $u_i \in L_p(M_i, m_i), i \in \mathbb{N}$, with

$$u_i \to u \quad \text{ in } L_p.$$

Proof. Let u and $(u_i)_{i\in\mathbb{N}}$ be given as above. Let us first consider the case where all the u_i are uniformly bounded, say $|u_i| \leq \beta$. Put $\alpha := \liminf_{i\to\infty} \mathcal{E}_N^{i,p}(u_i)$. Without restriction, we may assume $\alpha = \lim_{i\to\infty} \mathcal{E}_N^{i,p}(u_i)$. (Otherwise, we pass to a suitable subsequence.) To simplify the presentation we also assume $K = 0$. Since M is compact, it suffices to choose $\varphi = 1$. Then for all $r > 0$ and all (sufficiently small) $\epsilon > 0$

$$\begin{aligned}
\mathcal{E}_{0,N}^{p,r}(u, 1) &= \frac{N+p}{r^{N+p}} \int_M \int_M |u(x) - u(y)|^p \cdot 1_{\{\mathsf{d}(x,y)<r\}} m(dy)\, m(dx) \\
&\leq \frac{N+p}{r^{N+p}} \int_{M_i \times M} \int_{M_i \times M} \left[|u(x) - u_i(x')| + |u_i(x') - u_i(y')| + |u_i(y') - u(y)|\right]^p \\
&\qquad\qquad \cdot 1_{\{\mathsf{d}(x,y)<r,\ \hat{\mathsf{d}}_i(x,x')<\epsilon r/2,\ \hat{\mathsf{d}}_i(y,y')<\epsilon r/2\}} dq_i(y', y)\, dq_i(x', x) \\
&\quad + 2\frac{N+p}{r^{N+p}} (2\beta)^p \cdot q_i\Big(\{(x', x) \in M_i \times M : \hat{\mathsf{d}}_i(x', x) \geq \epsilon r/2\}\Big) \\
&\leq (1+\epsilon) \cdot \frac{N+p}{r^{N+p}} \int_{M_i} \int_{M_i} \left|u_i(x') - u_i(y')\right|^p \cdot 1_{\{\mathsf{d}_i(x',y')<(1+\epsilon)r\}} m_i(dy')\, m_i(dx') \\
&\quad + \frac{C(N, p, \epsilon)}{r^{N+p}} \int_{M_i \times M} \left|u_i(x') - u(x)\right|^p \, dq_i(x', x) \\
&\quad + 2\frac{N+p}{r^{N+p}} (2\beta)^p \cdot \frac{4}{\epsilon^2 r^2} \mathbb{D}^2\left((M_i, \mathsf{d}_i, m_i), (M, \mathsf{d}, m)\right).
\end{aligned}$$

For fixed r and ϵ and sufficiently large i the second and third term on the RHS are arbitrarily small. The first term can be estimated by

$$(1+\epsilon)^{1+N+p} \cdot \mathcal{E}_{0,N}^{i,p,(1+\epsilon)r}(u_i) \leq (1+\epsilon)^{1+N+p} \cdot \alpha.$$

Thus $\mathcal{E}_N^p(u) \leq \alpha$. This proves the claim for uniformly bounded sequences of u_i.
Now let an arbitrary sequence of u_i be given with $u_i \to u$ in L_p. For each $\beta > 0$ define $u_i^{(\beta)} := \min(\max(u_i, -\beta), \beta)$ and similarly $u^{(\beta)}$. Then $u_i^{(\beta)} \to u^{(\beta)}$ in L_p as $i \to \infty$ (for each β) and $\mathcal{E}_N^{i,p}(u_i^{(\beta)}) \nearrow \mathcal{E}_N^{i,p}(u_i)$ as $\beta \to \infty$ (for each i).
Hence, $\alpha = \liminf_{i\to\infty} \mathcal{E}_N^{i,p}(u_i)$ implies $\liminf_{i\to\infty} \mathcal{E}_N^{i,p}(u_i^{(\beta)}) \leq \alpha$ for each β. By the first part of the proof, it therefore follows $\mathcal{E}_N^p(u^{(\beta)}) \leq \alpha$ for each β. This finally yields $\mathcal{E}_N^p(u) \leq \alpha$. $\square$

Of particular interest is the case $p = 2$. It allows to define a Laplace operator on $L_2(M, m)$. The definition depends on the number N from the measure contraction property $\mathsf{MCP}(K, N)$. On the other hand, everything will be independent of the choice of K.

Theorem 6.4. *(i) The 2-nd order energy integral $\mathcal{E}_N^2$ extends to a bilinear form*

$$\mathcal{E}_N(u, v) := \frac{1}{4}\mathcal{E}_N^2(u+v) - \frac{1}{4}\mathcal{E}_N^2(u-v)$$

with domain $W^{1,2}_N(M)$. This is a strongly local Dirichlet form on $L_2(M,m)$ (not necessarily densely defined).
(ii) There exists a unique linear operator Δ_N with domain $\mathcal{D}(\Delta_N) \subset W^{1,2}_N(M) \subset L_2(M,m)$ such that

$$\mathcal{E}_N(u,v) = -\int (\Delta_N u)\cdot v\, dm \tag{6.4}$$

for all $v \in W^{1,2}_N(M)$ and $u \in \mathcal{D}(\Delta_N)$.
(iii) For each compact $K \subset M$ there exists a constant $C > 0$ such that for all $x \in \mathrm{supp}[m]$, all r with $B_r(x) \subset K$ and all $u \in W^{1,2}_N(M)$

$$\int_{B_{3r}(x)} \mu_N(u,dy) \geq \frac{C}{r^2}\int_{B_r(x)} |u(y)-\overline{u}_{r,x}|^2\, dm(y).$$

Here $\mu_N(u,.)$ denotes the energy measure of u associated with $\mathcal{E}_N$ and $\overline{u}_{r,x} = \frac{1}{m(B_r(x))}\int_{B_r(x)} u\, dm$.

Proof. (i) and (ii) are obvious. In order to prove (iii) we follow the argumentation in [St98], Theorem 6.3 and Corollary 6.4. Assume without restriction that $K = B_R(x_0)$. Choose a continuous approximation φ of the indicator function $1_{B_r(x)}$ and choose $\eta(s) = (p+N)\cdot s^{p+N-1}\cdot 1_{[0,1]}(s)$. Then $\varphi_{2r} \leq 2\cdot 1_{B_{3r}(x)}$ and thus by the Subpartitioning Lemma

$$\begin{aligned}
2\int_{B_{3r}(x)} d\mu_N(u,dy) &\geq \frac{2+N}{(2r)^{2+N}}\int_{B_r(x)}\int_{B_{2r}(y)} |u(y)-u(z)|^2\cdot\left(\frac{\mathsf{d}(y,z)}{s_{K,N}(\mathsf{d}(y,z))}\right)^{N-1} dm(z)\, dm(y)\\
&\geq \frac{C}{r^{2+N}}\int_{B_r(x)}\int_{B_r(x)} |u(y)-u(z)|^2\, dm(z)\, dm(y)\\
&= \frac{2C}{r^2}\cdot\frac{m(B_r(x))}{r^N}\int_{B_r(x)} |u(y)-\overline{u}_{r,x}|^2\, dm(y)\\
&\geq \frac{2C}{r^2}\cdot\frac{m(B_R(x_0))}{(2R)^N}\int_{B_r(x)} |u(y)-\overline{u}_{r,x}|^2\, dm(y).
\end{aligned}$$

□

Recall that according to the Bishop-Gromov volume growth inequality (2.5) the limit

$$\omega_N(x) := \lim_{r\to 0}\frac{m(B_r(x))}{r^N}$$

exists for each $x \in \mathrm{supp}[m]$. Moreover, it is positive and lower semicontinuous on $\mathrm{supp}[m]$. Assume for the sequel

(*) $\quad$ *$M = \mathrm{supp}[m]$ and ω_N is locally bounded on M.*

Note that $\omega_N \equiv \infty$ on $\mathrm{supp}[m]$ as soon as (M,d,m) also satisfies the measure contraction property $\mathsf{MCP}(K',N')$ for some $N' < N$ (and some K'). Hence, there exists at most one number N for which the above assumption is fulfilled.
Define 'the Dirichlet form' $(\mathcal{E}, W^{1,2}_0(M))$ on $L_2(M,m)$ as the closure of $(\mathcal{E}, \mathcal{C}^{Lip}_c(M))$ where

$$\mathcal{E}(u,u) := \int_M \frac{1}{\omega_N(x)}\,\mu_N(u,dx). \tag{6.5}$$

Moreover, define 'the Laplace operator' on M as the linear operator Δ with domain $\mathcal{D}(\Delta) \subset W^{1,2}_0(M) \subset L_2(M,m)$ such that $\mathcal{E}(u,v) = -\int(\Delta u)\cdot v\, dm$ for all $v \in W^{1,2}_0(M)$ and $u \in \mathcal{D}(\Delta)$.

Example 6.5. In the case of a Riemannian manifold M as considered in Corollary 5.5(i), the operator Δ coincides with the Laplace-Beltrami operator on $L_2(M,m)$.

Corollary 6.6. *For each metric measure space satisfying* $\mathsf{MCP}(K,N)$ *and condition* $(*)$*:*

(i) $(\mathcal{E}, W_0^{1,2}(M))$ *is a strongly local, regular Dirichlet form on* $L_2(M,m)$*, densely defined with core* $\mathcal{C}_c^{Lip}(M)$*.*

(ii) *A scale invariant Poincaré inequality (in the same form as in the previous theorem) holds true for* $\mathcal{E}$*.*

(iii) *For each* $u \in \mathcal{C}_{loc}^{Lip}(M)$

$$\mu(u,dx) \le N \cdot |\nabla u|^2(x) \cdot m(dx) \tag{6.6}$$

where $|\nabla u|(x) := \limsup_{y\to x} \frac{|u(x)-u(y)|}{\mathsf{d}(x,y)}$ *denotes the 'length of the gradient' of* u*. Moreover, for each* $z \in M$ *and* $u = \mathsf{d}(z,.)$ *we have* $|\nabla u|(.) \equiv 1$ *and*

$$\mu(u,dx) \ge \frac{N+2}{2^{3N+2}}\, m(dx).$$

Proof. Properties (i) and (ii) are obvious due to the previous theorem. In order to see the first assertion of (iii), note that by monotone convergence for all nonnegative continuous φ

$$\begin{aligned}
\int_M |\nabla u|^2(x)\varphi(x)\, m(dx) &= \lim_{r\to 0} \int_M \sup_{y\in B_r(x)} \left|\frac{u(x)-u(y)}{\mathsf{d}(x,y)}\right|^2 \cdot \frac{m(B_r(x))}{r^N} \frac{\varphi(x)}{\omega_N(x)}\, m(dx) \\
&\ge \lim_{r\to 0} \frac{1}{r^N} \int_M \int_{B_r(x)} \left|\frac{u(x)-u(y)}{\mathsf{d}(x,y)}\right|^2 dm(y) \frac{\varphi(x)}{\omega_N(x)}\, m(dx) \\
&= \frac{1}{N} \int \frac{\varphi(x)}{\omega_N(x)}\, \mu_N(u,dx) = \frac{1}{N}\int \varphi(x)\, \mu(u,dx).
\end{aligned}$$

For the second assertion of (iii), let $z,x \in M$ and $r>0$ be given with $r \le \mathsf{d}(x,z)$. Choose a point ξ on a geodesic connecting x and z with $\mathsf{d}(x,\xi) = \frac{3}{4}r$. Then $|u(x)-u(y)| \ge r/2$ for all $y \in B_{r/4}(\xi)$. Hence,

$$\begin{aligned}
\frac{1}{N+2}\int \varphi(x)\,\mu(u,dx) &= \lim_{r\to 0} \frac{1}{r^{N+2}} \int_M \int_{B_r(x)} |u(x)-u(y)|^2\, m(dy) \frac{\varphi(x)}{\omega_N(x)}\, m(dx) \\
&\ge \lim_{r\to 0} \frac{1}{r^{N+2}} \int_M m(B_{r/4}(\xi)) \cdot \left(\frac{r}{2}\right)^2 \frac{\varphi(x)}{\omega_N(x)}\, m(dx) \\
&\ge \lim_{r\to 0} \frac{1}{4r^N 8^N} \int_M m(B_{2r}(\xi)) \cdot \frac{\varphi(x)}{\omega_N(x)}\, m(dx) \ge \frac{1}{2^{3N+2}} \int_M \varphi(x)\, m(dx).
\end{aligned}$$

□

Extending fundamental work of De Giorgi, Nash, Moser, Grigor'yan, Saloff-Coste and many others, it was shown in [St96] that

(i) a doubling property for the volume of balls (as in Cor. 2.4),

(ii) a scale-invariant Poincaré inequality on balls (as in Thm. 6.4), and

(iii) a gradient estimate (as in (6.6)) for cut-off functions allow to deduce a variety of regularity results in the general framework of strongly local, regular Dirichlet forms on locally compact metric spaces, generalizing classical regularity theory for second order elliptic operators on Euclidean or Riemannian spaces. In particular, as an immediate consequence of Corollary 6.6 and Corollary 2.4 we obtain

- **Parabolic Harnack Inequality** *for local weak solutions of the equation* $(\frac{1}{2}\Delta - \frac{\partial}{\partial t})u = 0$ *on* $\mathbb{R} \times M$*;*
- **Hölder Continuity** *for functions* u *as above and, in particular, for harmonic functions on* M*;*
- **Feller Property** *of the transition semigroup* $T_t := \exp(t\Delta/2)$ *on* $L_2(M, m)$ *resp. on* $\mathcal{C}_0(M)$*;*
- **Gaussian Estimates** *for the heat kernel* $p_t(x, y)$*, i.e. for the density of the transition semigroup* T_t*.*

We refer to [St96] and [St98] for detailed formulations of these properties as well as for proofs. Instead of going into technicalities we will close here, presenting a simple nice application.

Corollary 6.7. *The measure contraction property* $\mathsf{MCP}(0, 2)$ *for* (M, d, m) *together with assumption (*) for* $N = 2$ *imply that the Dirichlet form* $(\mathcal{E}, W_0^{1,2}(M))$ *is recurrent.*
That is, every nonnegative superharmonic function on M *must be constant. Or in other words, the space* M *is 'parabolic'.*

Remark 6.8. Defining the p-th order energy integral and the Dirichlet form on $L_2(M, m)$, we followed our previous approach in [St98]. The latter provided the first construction of a Dirichlet form on a metric measure space (M, d, m). Later on, Cheeger [Ch99] presented a remarkable alternative construction, based on the concept of upper gradients. According to Corollary 7.6(iii) the measure contraction property also implies the Poincaré inequality in Cheeger's context.
There is a huge literature on Sobolev spaces over metric measure spaces, starting with the work of Hajlasz [Ha96]. See [He01], [AT04] and references therein.
After this paper was completed, we learned of related work by Ohta [Oh05].[4]

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