A Curvature-Dimension Condition for Metric Measure Spaces

Une Condition de Type Courbure-Dimension pour des Espaces Métriques Mesurés

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Abstract

We present a curvature-dimension condition $CD(K, N)$ for metric measure spaces (M, d, m). In some sense, it will be the geometric counterpart to the Bakry-Émery condition for Dirichlet forms [1]. For Riemannian manifolds, it holds if and only if $\dim(M) \le N$ *and* $\mathrm{Ric}_M(\xi, \xi) \ge K \cdot |\xi|^2$ for all $\xi \in TM$. The curvature bound from [6,7] is the limit case $CD(K, \infty)$.

Our curvature-dimension condition is stable under convergence, cf. [6,7]. Furthermore, it entails various geometric consequences e.g. the Bishop-Gromov theorem and the Bonnet-Myers theorem. In both cases, we obtain the sharp estimates known from the Riemannian case. *To cite this article: K. T. Sturm, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Nous présentons une condition de type courbure-dimension $CD(K, N)$ pour des espaces métriques mesurés (M, d, m), qui se peut regarder comme une contrepartie géométrique de laquelle de Bakry-Émery pour les formes Dirichlet [1]. Pour les varietés riemanniennes, elle est satisfaite si et seulement si $\dim(M) \le N$ *et* $\mathrm{Ric}_M(\xi, \xi) \ge K \cdot |\xi|^2$ pour tout $\xi \in TM$. La borne de la courbure de [6,7] est le cas limite $CD(K, \infty)$.

Notre condition est stable pour la convergence, selon [6,7]. Elle comporte des conséquences géométriques diverses, comme les théorèmes de Bishop-Gromov et de Bonnet-Myers. Dans les deux cas, on obtient des estimations optimales connues dans le cas riemannien. *Pour citer cet article : K. T. Sturm, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

A *metric measure space* will always be a triple (M, d, m) where (M, d) is a complete separable metric space and m is a locally finite measure on M equipped with its Borel σ-algebra. The case $m(M) = 0$ will be excluded. $\mathcal{P}_2(M, \mathsf{d})$ denotes the L_2-Wasserstein space of probability measures on M and d_W the corresponding L_2-Wasserstein distance. The subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(M, \mathsf{d}, m)$.

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Given a metric measure space (M, d, m) and a number $N \in \mathbb{R}$, $N \geq 1$ we define the *Rényi entropy functional* $S_N(.|m): \mathcal{P}_2(M, \mathsf{d}) \to \mathbb{R}$ with respect to m by

$$S_N(\nu|m) := -\int \rho^{-1/N} d\nu$$

where ρ denotes the density of the absolutely continuous part ν^c in the Lebesgue decomposition $\nu = \nu^c + \nu^s = \rho m + \nu^s$ of $\nu \in \mathcal{P}_2(M, \mathsf{d})$. Note that $S_1(\nu|m) = -m(\mathrm{supp}[\nu^c])$. The functional $\tilde{S}_N := N + N\,S_N$ shares various properties with the relative Shannon entropy $\mathrm{Ent}(.|m)$. For instance, if m is a probability measure then $\tilde{S}_N(.|m) \geq 0$ on $\mathcal{P}_2(M, \mathsf{d})$ and $\tilde{S}_N(\nu|m) = 0$ if and only if $\nu = m$. If $m(M)$ is finite then $\mathrm{Ent}(\nu|m) = \lim_{N\to\infty} N(1 + S_N(\nu|m))$ for each $\nu \in \mathcal{P}_2(M, \mathsf{d})$.

Definition 1 *Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we say that a metric measure space (M, d, m) satisfies the* curvature-dimension condition *$CD(K, N)$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exist an optimal coupling q of ν_0, ν_1 and a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0, ν_1 with*

$$S_{N'}(\Gamma(t)|m) \leq -\int_{M\times M} \left[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x_0, x_1)) \cdot \rho_0^{-1/N'}(x_0)\tau^{(t)}_{K,N'}(\mathsf{d}(x_0, x_1)) \cdot \rho_1^{-1/N'}(x_1)\right] dq(x_0, x_1)$$

for all $t \in [0, 1]$ and all $N' \geq N$. Here ρ_i denotes the density of the absolutely continuous part of ν_i w.r.t. m (for $i = 0, 1$) and for each $\theta \in \mathbb{R}_+$

$$\tau^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq (N-1)\pi^2 \\ t^{1/N}\left(\sin\left(\sqrt{\frac{K}{N-1}}\,t\theta\right) \Big/ \sin\left(\sqrt{\frac{K}{N-1}}\,\theta\right)\right)^{1-1/N}, & \text{if } 0 < K\theta^2 < (N-1)\pi^2 \\ t, & \text{if } K\theta^2 = 0 \text{ or} \\ & \text{if } K\theta^2 < 0 \text{ and } N = 1 \\ t^{1/N}\left(\sinh\left(\sqrt{\frac{-K}{N-1}}\,t\theta\right) \Big/ \sinh\left(\sqrt{\frac{-K}{N-1}}\,\theta\right)\right)^{1-1/N}, & \text{if } K\theta^2 < 0 \text{ and } N > 1. \end{cases}$$

Theorem 2 *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and let numbers $K, N \in \mathbb{R}$ with $N \geq 1$ be given.*

(i) *The metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(K, N)$ if and only if the Riemannian manifold M has Ricci curvature $\geq K$ and dimension $\leq N$.*

(ii) *Moreover, in this case for every measurable function $V : M \to \mathbb{R}$ the weighted space $(M, \mathsf{d}, V\,m)$ satisfies the curvature-dimension condition $CD(K + K', N + N')$ provided*

$$\mathrm{Hess}\, V^{1/N'} \leq -\frac{K'}{N'} \cdot V^{1/N'}$$

for some numbers $K' \in \mathbb{R}$, $N' > 0$ in the following sense:

$$V(\gamma_t)^{1/N'} \geq \sigma^{(1-t)}_{K',N'}(\mathsf{d}(\gamma_0, \gamma_1))\, V(\gamma_0)^{1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(\gamma_0, \gamma_1))\, V(\gamma_1)^{1/N'}$$

for each geodesic $\gamma : [0, 1] \to M$ and each $t \in [0, 1]$. Here $\sigma^{(t)}_{K',N'}(\theta) := t^{-1/N'} \cdot \tau^{(t)}_{K',N'+1}(\theta)^{1+1/N'}$.

This essentially follows from estimates for the Jacobian of transport maps in [3] and [5]. The particular case of the $CD(0, N)$ condition has already been treated in [5] and later independently in [4].

Let us have a closer look on these results if M is a subset of the real line equipped with the usual distance d and the 1-dimensional Lebesgue measure m.

Example 1 ***(i)*** *For each pair of real numbers $K>0, N>1$ the space $([0,L],\mathsf{d},V\,m)$ with $L:=\sqrt{\frac{N-1}{K}}\pi$ and $V(x)=\sin\left(\sqrt{\frac{K}{N-1}}\,x\right)^{N-1}$ satisfies the curvature-dimension condition $CD(K,N)$.*

(ii) *For each pair of real numbers $K\le 0, N>1$ the space $(\mathbb{R}_+,\mathsf{d},V\,m)$ with $V(x)=\sinh\left(\sqrt{\frac{-K}{N-1}}\,x\right)^{N-1}$, if $K<0$, and $V(x)=x^{N-1}$, if $K=0$, satisfies the curvature-dimension condition $CD(K,N)$.*

(iii) *For each pair of real numbers $K<0, N>1$ the space $(\mathbb{R},\mathsf{d},V\,m)$ with $V(x)=\cosh\left(\sqrt{\frac{-K}{N-1}}\,x\right)^{N-1}$ satisfies the curvature-dimension condition $CD(K,N)$.*

Note that for $N\to\infty$ the weight V from example (iii) from above converges to the weight $V(x)=\exp\left(\frac{-K}{2}\,x^2\right)$. Also note that according to [2], the examples (i)-(iii) equipped with natural weighted Laplacians are also the prototypes for the Bakry-Émery curvature-dimension condition.

Proposition 3 ('Generalized Brunn-Minkowski Inequality') *Assume that the metric measure space (M,d,m) satisfies the curvature-dimension condition $CD(K,N)$ for some real numbers $K,N\in\mathbb{R}$, $N\ge 1$. Then for all measurable sets $A_0,A_1\subset M$ with $m(A_0)\cdot m(A_1)>0$, all $t\in[0,1]$ and all $N'\ge N$*

$$m(A_t)^{1/N'}\ge\tau_{K,N'}^{(1-t)}(\Theta)\cdot m(A_0)^{1/N'}+\tau_{K,N'}^{(t)}(\Theta)\cdot m(A_1)^{1/N'}$$

where A_t denotes the set of points γ_t on geodesics with endpoints $\gamma_0\in A_0$ and $\gamma_1\in A_1$ and where $\Theta=\inf_{x_0\in A_0,x_1\in A_1}\mathsf{d}(x_0,x_1)$ if $K\ge 0$ and $\Theta=\sup_{x_0\in A_0,x_1\in A_1}\mathsf{d}(x_0,x_1)$ if $K<0$. In particular, if $K\ge 0$ then

$$m(A_t)^{1/N'}\ge(1-t)\cdot m(A_0)^{1/N'}+t\cdot m(A_1)^{1/N'}.$$

Now let us fix a point $x_0\in\operatorname{supp}[m]$ and study the growth of the volume of concentric balls as well as the growth of the volume of the corresponding spheres:

$$v(r):=m(\overline{B}_r(x_0)),\qquad s(r):=\limsup_{\delta\to 0}\frac{1}{\delta}\cdot m\left(\overline{B}_{r+\delta}(x_0)\setminus B_r(x_0)\right).$$

Theorem 4 ('Generalized Bishop-Gromov Volume Growth Inequality') *Assume that the metric measure space (M,d,m) satisfies the curvature-dimension condition $CD(K,N)$ for some $K,N\in\mathbb{R}$, $N\ge 1$. Then each bounded set $M'\subset M$ has finite volume. Moreover, either m is supported by one point or all points and all spheres have mass 0.*

More precisely, if $N>1$ then for each fixed $x_0\in\operatorname{supp}[m]$ and all $0<r<R\le\sqrt{\frac{N-1}{K\vee 0}}\cdot\pi$

$$\frac{s(r)}{s(R)}\ge\left(\frac{\sin\left(\sqrt{\frac{K}{N-1}}r\right)}{\sin\left(\sqrt{\frac{K}{N-1}}R\right)}\right)^{N-1}\qquad\text{and}\qquad\frac{v(r)}{v(R)}\ge\frac{\int_0^r\sin\left(\sqrt{\frac{K}{N-1}}t\right)^{N-1}dt}{\int_0^R\sin\left(\sqrt{\frac{K}{N-1}}t\right)^{N-1}dt}$$

with $s(.)$ and $v(.)$ defined as above and with the usual interpretation of the RHS if $K\le 0$. In particular, if $K=0$

$$\frac{s(r)}{s(R)}\ge\left(\frac{r}{R}\right)^{N-1}\qquad\text{and}\qquad\frac{v(r)}{v(R)}\ge\left(\frac{r}{R}\right)^{N}.$$

The latter also holds true if $N=1$ and $K\le 0$.

For each K and each *integer* $N>1$ the simply connected spaces of dimension N and constant curvature $K/(N-1)$ provide examples where these volume growth estimates are sharp. But also for arbitrary *real* numbers $N>1$ these estimates are sharp as demonstrated by Example 1(i) and (ii) where equality is attained.

Corollary 5 ('Doubling') *For each metric measure space (M, d, m) which satisfies the curvature-dimension condition $CD(K,N)$ for some $K, N \in \mathbb{R}$, $N \geq 1$, the doubling property holds on each bounded subset $M' \subset \text{supp}[m]$. In particular, each bounded closed subset $M' \subset \text{supp}[m]$ is compact.*

If $K \geq 0$ or $N = 1$ the doubling constant is $\leq 2^N$. Otherwise, it can be estimated by $2^N \cdot \cosh\left(\sqrt{\frac{-K}{N-1}}\, L\right)^{N-1}$ where L is the diameter of M'.

Corollary 6 ('Hausdorff Dimension') *Each metric measure space (M, d, m) which satisfies the curvature-dimension condition $CD(K,N)$ for some $K, N \in \mathbb{R}$, $N \geq 1$, has Hausdorff dimension $\leq N$.*

Corollary 7 ('Generalized Bonnet-Myers Theorem') *For every metric measure space (M, d, m) which satisfies the curvature-dimension condition $CD(K,N)$ for some real numbers $K > 0$ and $N \geq 1$ the support of m is compact and has diameter*

$$L \leq \sqrt{\frac{N-1}{K}}\, \pi.$$

In particular, if $K > 0$ and $N = 1$ then $\text{supp}[m]$ consists of one point.

Theorem 8 ('Stability under Convergence) *Let $((M_n, \mathsf{d}_n, m_n))_{n \in \mathbb{N}}$ be a sequence of normalized metric measure spaces where for each $n \in \mathbb{N}$ the space (M_n, d_n, m_n) satisfies the curvature-dimension condition $CD(K_n, N_n)$ and has diameter $\leq L_n$. Assume that for $n \to \infty$*

$$(M_n, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

and $(K_n, N_n, L_n) \to (K, N, L)$ for some triple $(K, N, L) \in \mathbb{R}^2$ satisfying $K \cdot L^2 < (N-1)\pi^2$. Then the space (M, d, m) satisfies the curvature-dimension condition $CD(K,N)$ and has diameter $\leq L$.

Corollary 9 ('Compactness') *For each triple $(K, L, N) \in \mathbb{R}^3$ with $K \cdot L^2 < (N-1)\pi^2$ the family $\mathbb{X}_1(K, N, L)$ of isomorphism classes of normalized metric measure spaces which satisfy the curvature-dimension condition $CD(K,N)$ and which have diameter $\leq L$ is compact w.r.t. $\mathbb{D}$.*

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