Generalized Ricci Bounds and Convergence of Metric Measure Spaces

Bornes Généralisées de la Courbure Ricci et Convergence des Espaces Métriques Mesurés

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Abstract

We introduce and analyze curvature bounds $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ for metric measure spaces (M, d, m), based on convexity properties of the relative entropy $\mathrm{Ent}(.|m)$. For Riemannian manifolds, $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ if and only if $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ for all $\xi \in TM$.

We define a complete separable metric $\mathbb{D}$ on the family of all isomorphism classes of normalized metric measure spaces. It has a natural interpretation in terms of mass transportation. Our lower curvature bounds are stable under $\mathbb{D}$-convergence. We also prove that the family of normalized metric measure spaces with doubling constant $\leq C$ is closed under $\mathbb{D}$-convergence. Moreover, the subfamily of spaces with diameter $\leq R$ is compact. *To cite this article: K. T. Sturm, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Nous introduisons et étudions bornes de la courbure $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ pour des espaces métriques mesurés (M, d, m), basées sur les propriétés de convexité de l'entropie relative $\mathrm{Ent}(.|m)$. Pour les varietés riemanniennes, $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) \geq K$ si et seulement si $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ pour tous $\xi \in TM$.

Nous définissons une métrique $\mathbb{D}$ complete separable sur la famille des classes d'isomorphie des espaces métrique mesurés normalisés. Elle a une naturel interprétation dans le contexte du transport de masse. Nos bornes inférieures de la courbure sont stables pour la $\mathbb{D}$-convergence. Nous démontrons aussi que pour la $\mathbb{D}$-convergence la famille des espaces métriques mesurés normalisés avec la constante de doublement $\leq C$ est fermée et, en plus, la sous-famille avec les diamètres $\leq R$ est compacte. *Pour citer cet article : K. T. Sturm, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

A *metric measure space* is a triple (M, d, m) where (M, d) is a complete separable metric space and m is a measure on M (equipped with its Borel σ-algebra $\mathcal{B}(M)$) which is locally finite in the sense that

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$m(B_r(x)) < \infty$ for all $x \in M$ and all sufficiently small $r > 0$. A metric measure space (M, d, m) is called *normalized* iff $m(M) = 1$. Two metric measure spaces (M, d, m) and (M', d', m') are called *isomorphic* iff there exists an isometry $\psi : M_0 \to M_0'$ between the supports $M_0 := \mathrm{supp}[m] \subset M$ and $M_0' := \mathrm{supp}[m'] \subset M'$ such that $m' = \psi_* m$. The *diameter* of a metric measure space is given by $\mathrm{diam}(M, \mathsf{d}, m) := \sup\{d(x,y) : x, y \in \mathrm{supp}[m]\}$. Its *variance* is defined as $\mathbb{V}ar(M, \mathsf{d}, m) := \inf \int_{M'} \mathsf{d}'^2(z,x) dm'(x)$ where the inf is taken over all metric measure spaces (M', d', m') which are isomorphic to (M, d, m) and over all $z \in M'$. The family of all isomorphism classes of normalized metric measure spaces with finite variances will be denoted by $\mathbb{X}_1$, cf. [3], chapter $3\frac{1}{2}$.

Definition 1 The *distance* $\mathbb{D}$ between two metric measure spaces is defined by

$$\mathbb{D}((M, \mathsf{d}, m), (M', \mathsf{d}', m')) = \inf_{\hat{\mathsf{d}}, \hat{m}} \left(\int_{M \sqcup M'} \hat{\mathsf{d}}^2(x,y) d\hat{m}(x,y) \right)^{1/2}$$

where the infimum is taken over all couplings $\hat{\mathsf{d}}$ of d, d' and over all couplings $\hat{m}$ of m, m' Here a measure $\hat{m}$ on the product space $M \times M'$ is called *coupling of* m *and* m' iff $\hat{m}(A \times M') = m(A)$ and $\hat{m}(M \times A') = m'(A')$ for all measurable sets $A \subset M$, $A' \subset M'$; a pseudo metric $\hat{\mathsf{d}}$ on the disjoint union $M \sqcup M'$ is called *coupling of* d *and* d' iff $\hat{\mathsf{d}}(x,y) = \mathsf{d}(x,y)$ and $\hat{\mathsf{d}}(x',y') = \mathsf{d}'(x',y')$ for all $x, y \in \mathrm{supp}[m] \subset M$ and all $x', y' \in \mathrm{supp}[m'] \subset M'$.
One easily verifies that

$$\mathbb{D}((M, \mathsf{d}, m), (M', \mathsf{d}', m')) = \inf \hat{\mathsf{d}}_W(\psi_* m, \psi'_* m')$$

where the infimum is taken over all metric spaces $(\hat{M}, \hat{\mathsf{d}})$ with isometric embeddings $\psi : M_0 \hookrightarrow \hat{M}$, $\psi' : M_0' \hookrightarrow \hat{M}$ of the supports M_0 and M_0' of m and m', resp., and where $\hat{d}_W$ denotes the L_2-Wasserstein distance between probability measures on $(\hat{M}, \hat{\mathsf{d}})$. In particular, if $(M, \mathsf{d}) = (M', \mathsf{d}')$ then $\mathbb{D}((M, \mathsf{d}, m), (M, \mathsf{d}, m')) \le \mathsf{d}_W(m, m')$. In general, however, there will be no equality.

Theorem 2 $(\mathbb{X}_1, \mathbb{D})$ *is a complete separable length metric space.*

We say that a metric measure space (M, d, m) has the *restricted doubling property with doubling constant* C iff $m(B_{2r}(x)) \le C \cdot m(B_r(x))$ for all $x \in \mathrm{supp}[m]$ and all $r > 0$.

Theorem 3 *The restricted doubling property is stable under* $\mathbb{D}$*-convergence. That is, if for all* $n \in \mathbb{N}$ *the normalized metric measure spaces* (M_n, d_n, m_n) *have the restricted doubling property with a common doubling constant* C *and if* $(M_n, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$ *as* $n \to \infty$ *then also* (M, d, m) *has the restricted doubling property with the same constant* C.

Theorem 4 *('Compactness')*
For each pair $(C, R) \in \mathbb{R}_+ \times \mathbb{R}_+$ *the family* $\mathbb{X}_1(C, R)$ *of all isomorphism classes of normalized metric measure spaces with ('restricted') doubling constant* $\le C$ *and diameter* $\le R$ *is compact under* $\mathbb{D}$*-convergence.*

Given a metric measure space (M, d, m) we denote by $\mathcal{P}_2(M, \mathsf{d})$ the space of all probability measures ν on M with $\int_M \mathsf{d}^2(o,x) d\nu(x) < \infty$ for some (hence all) $o \in M$. The L_2-*Wasserstein distance* on $\mathcal{P}_2(M, \mathsf{d})$ is defined by $\mathsf{d}_W(\mu, \nu) = \inf(\int_{M \times M} \mathsf{d}^2(x,y) dq(x,y))^{1/2}$ with the infimum taken over all couplings q of μ and ν, see [7]. For $\nu \in \mathcal{P}_2(M, \mathsf{d})$ we define the *relative entropy w.r.t.* m by

$$\mathrm{Ent}(\nu|m) := \lim_{\epsilon \searrow 0} \int_{\{\rho > \epsilon\}} \rho \log \rho \, dm$$

if ν is absolutely continuous w.r.t. m with density $\rho = \frac{d\nu}{dm}$ and by $\mathrm{Ent}(\nu|m) := +\infty$ if ν is singular w.r.t. m. Finally, we put $\mathcal{P}_2^*(M, \mathsf{d}, m) := \{\nu \in \mathcal{P}_2(M, \mathsf{d}) : \ \mathrm{Ent}(\nu|m) < +\infty\}$.

Lemma 5 *If m has finite mass, then* $\mathrm{Ent}(\cdot\,|m)$ *is lower semicontinuous and* $\neq -\infty$ *on* $\mathcal{P}_2(M, \mathsf{d})$.

Definition 6 (i) Given any number $K \in \mathbb{R}$, we say that a metric measure space (M, d, m) has *curvature* $\geq K$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_2^*(M, \mathsf{d}, m)$ connecting ν_0 and ν_1 with

$$\mathrm{Ent}(\Gamma(t)|m) \leq (1-t)\mathrm{Ent}(\Gamma(0)|m) + t\mathrm{Ent}(\Gamma(1)|m) - \frac{K}{2} t(1-t)\, \mathsf{d}_W^2(\Gamma(0), \Gamma(1))$$

for all $t \in [0,1]$. Moreover, we put $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) := \sup\{K \in \mathbb{R} : (M, \mathsf{d}, m) \text{ has curvature } \geq K.\}$

(ii) We say that a metric measure space (M, d, m) has *curvature* $\geq K$ *in the lax sense* iff for each $\epsilon > 0$ and for each pair $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ there exists an $\eta \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ with $\mathsf{d}_W(\eta, \nu_i) \leq \frac{1}{2}\, \mathsf{d}_W(\nu_0, \nu_1) + \epsilon$ for each $i = 0, 1$ and

$$\mathrm{Ent}(\eta|m) \leq \frac{1}{2}\mathrm{Ent}(\nu_0|m) + \frac{1}{2}\mathrm{Ent}(\nu_1|m) - \frac{K}{8}\, \mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon.$$

We denote the maximal K with this property by $\underline{\mathbb{C}\mathrm{urv}}_{lax}(M, \mathsf{d}, m)$.

(iii) We say that a metric measure space (M, d, m) has *locally curvature* $\geq K$ if each point of M has a neighborhood M' such that (M', d, m) has curvature $\geq K$. The maximal K with this property will be denoted by $\underline{\mathbb{C}\mathrm{urv}}_{loc}(M, \mathsf{d}, m)$.

Let us consider these curvature bounds under some of the **Basic Transformations**:

ISOMORPHISMS. $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) = \underline{\mathbb{C}\mathrm{urv}}(M', \mathsf{d}', m')$ for each (M', d', m') isomorphic to (M, d, m);

SCALING. $\underline{\mathbb{C}\mathrm{urv}}(M, \alpha\, \mathsf{d}, \beta m) = \alpha^{-2}\, \underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m)$ for all $\alpha, \beta > 0$;

WEIGHTS. $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, e^{-V} m) \geq \underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) + \underline{\mathrm{Hess}} V$ for each lower bounded, measurable function $V : M \to \mathbb{R}$ where $\underline{\mathrm{Hess}} V := \sup\{K \in \mathbb{R} : V \text{ is K-convex on } \mathrm{supp}[m]\}$;

SUBSETS. $\underline{\mathbb{C}\mathrm{urv}}(M', \mathsf{d}, m) \geq \underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m)$ for each convex $M' \subset M$;

PRODUCTS. $\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) = \inf_{i \in \{1,\ldots,l\}} \underline{\mathbb{C}\mathrm{urv}}(M_i, \mathsf{d}_i, m_i)$ if $(M, \mathsf{d}, m) = \bigotimes_{i=1}^l (M_i, \mathsf{d}_i, m_i)$ and if M is nonbranching and compact.

Here a metric space (M, d) is called *nonbranching* iff for each quadruple of points z, x_0, x_1, x_2 with z being the midpoint of x_0 and x_1 as well as the midpoint of x_0 and x_2 it follows that $x_1 = x_2$.

Theorem 7 *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and put $m' = e^{-V} m$ with a $\mathcal{C}^2$ function $V : M \to \mathbb{R}$. Then*

$$\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m') = \inf\left\{\mathrm{Ric}_M(\xi, \xi) + \mathrm{Hess} V(\xi, \xi) : \xi \in TM, |\xi| = 1\right\}.$$

In particular, (M, d, m) has curvature $\geq K$ if and only if the Ricci curvature of M is $\geq K$.

See [4] for the case $V = 0$ or [5] for the general case. Note that in the above Riemannian setting for each pair of points ν_0, ν_1 in $\mathcal{P}_2(M, \mathsf{d}, m)$ there exists a *unique* geodesic connecting them [1].

Lemma 8 *If M is compact then*

$$\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) = \underline{\mathbb{C}\mathrm{urv}}_{lax}(M, \mathsf{d}, m).$$

Of fundamental importance is that our curvature bounds for metric measure spaces are stable under convergence and that local curvature bounds imply global curvature bounds. The latter is in the spirit of the Globalization Theorem of Topogonov for lower curvature bounds (in the sense of Alexandrov) for metric spaces.

Theorem 9 *Let (M, d, m) be a compact, nonbranching metric measure space such that $\mathcal{P}_2^*(M, \mathsf{d}, m)$ is a geodesic space. Then*

$$\underline{\mathbb{C}\mathrm{urv}}(M, \mathsf{d}, m) = \underline{\mathbb{C}\mathrm{urv}}_{loc}(M, \mathsf{d}, m).$$

Theorem 10 *Let $((M_n,\, \mathsf{d}_n, m_n))_{n\in\mathbb{N}}$ be a sequence of normalized metric measure spaces with uniformly bounded diameter and with $(M_n,\, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M,\, \mathsf{d}, m)$. Then*

$$\limsup_{n\to\infty} \underline{\mathbb{C}\mathrm{urv}}_{lax}(M_n,\, \mathsf{d}_n, m_n) \le \underline{\mathbb{C}\mathrm{urv}}_{lax}(M,\, \mathsf{d}, m).$$

Example 1 Given any abstract Wiener space (M, H, m) define a pseudo metric on M by $\mathsf{d}(x, y) := \|x - y\|_H$ if $x - y \in H$ and $\mathsf{d}(x, y) := \infty$ else and consider the 'pseudo metric measure space' $(M,\, \mathsf{d}, m)$. Then

$$\underline{\mathbb{C}\mathrm{urv}}_{lax}(M,\, \mathsf{d}, m) = 1.$$

Of course, formally this does not fit in our framework. Nevertheless, the definition of the L_2-Wasserstein distance d_W derived from this pseudo metric d perfectly makes sense (cf. [2]) and also the relative entropy is well-defined.

Lower bounds for the curvature will imply upper estimates for the volume growth of concentric balls. In the Riemannian setting, this is the content of the famous Bishop-Gromov volume comparison theorem. In the general case (without any dimensional restriction) these estimates, however, have to take into account that the volume can grow much faster than exponentially. For instance, we already observe squared exponential volume growth if we equip the one-dimensional Euclidean space with the measure $dm(x) = \exp(-Kx^2/2)\, dx$ for some $K < 0$.

Theorem 11 *Let $(M,\, \mathsf{d}, m)$ be an arbitrary metric measure space with $\underline{\mathbb{C}\mathrm{urv}}(M,\, \mathsf{d}, m) \ge K$ for some $K \le 0$. For fixed $x \in \mathrm{supp}[m] \subset M$ consider the volume growth $v_R := m(\overline{B}_R(x))$ of closed balls centered at x. Then for all $R \ge 2\epsilon > 0$*

$$v_R \le v_{2\epsilon} \cdot (v_{2\epsilon}/v_\epsilon)^{R/\epsilon} \cdot \exp\left(|K|\,(R + \epsilon/2)^2/2\right).$$

In particular, each ball in M has finite volume.

Theorem 12 *If $\underline{\mathbb{C}\mathrm{urv}}(M,\, \mathsf{d}, m) \ge K \ge 0$ then for all $x \in M$ and for all $R \ge 3\epsilon > 0$ the volume of spherical shells $v_{R,\epsilon} := m(\overline{B}_R(x) \setminus B_{R-\epsilon}(x))$ can be estimated by*

$$v_{R,\epsilon} \le v_{3\epsilon} \cdot (v_{3\epsilon}/v_\epsilon)^{R/2\epsilon} \cdot \exp\left(-K\left[(R - 3\epsilon)^2 - \epsilon^2\right]/2\right).$$

In particular, $K > 0$ implies that m has finite mass and finite variance.

undefined

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