A Semigroup Approach to Harmonic Maps

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Dedicated to the memory of Professor Dr. Heinz Bauer

Abstract. We present a semigroup approach to harmonic maps between metric spaces. Our basic assumption on the target space (N, d) is that it admits a "barycenter contraction", i.e. a contracting map which assigns to each probability measure q on N a point $b(q)$ in N. This includes all metric spaces with globally nonpositive curvature in the sense of Alexandrov as well as all metric spaces with globally nonpositive curvature in the sense of Busemann. It also includes all Banach spaces.

The analytic input comes from the domain space (M, ρ) where we assume that we are given a Markov semigroup $(p_t)_{t>0}$. Typical examples come from elliptic or parabolic second order operators on $\mathbb{R}^n$, from Lévy type operators, from Laplacians on manifolds or on metric measure spaces and from convolution operators on groups. In contrast to the work of Korevaar, Schoen (1993, 1997), Jost (1994, 1997), Eells, Fuglede (2001) our semigroups are not required to be symmetric.

The linear semigroup acting e.g. on the space of bounded measurable functions $u : M \to \mathbb{R}$ gives rise to a nonlinear semigroup $(P_t^*)_t$ acting on certain classes of measurable maps $f : M \to N$. We will show that contraction and smoothing properties of the linear semigroup $(p_t)_t$ can be extended to the nonlinear semigroup $(P_t^*)_t$, for instance, L_p-L_q smoothing, hypercontractivity, and exponentially fast convergence to equilibrium. Among others, we state existence and uniqueness of the solution to the Dirichlet problem for harmonic maps between metric spaces. Moreover, for this solution we prove Lipschitz continuity in the interior and Hölder continuity at the boundary.

Our approach also yields a new interpretation of curvature assumptions which are usually required to deduce regularity results for the harmonic map flow: lower Ricci curvature bounds on the domain space are equivalent to estimates of the L_1-Wasserstein distance between the distribution of two Brownian motions in terms of the distance of their starting points; nonpositive sectional curvature on the target space is equivalent to the fact that the L_1-Wasserstein distance of two distributions always dominates the distance of their barycenters.

Keywords: harmonic map, barycenter, Markov semigroup, nonlinear Markov operator, NPC space, Hadamard space, Dirichlet problem, coupling, Wasserstein distance.

Introduction

A smooth map $f : M \to N$ between Riemannian manifolds is called harmonic iff its tension field $\tau(f) := \mathrm{trace}\nabla(df)$ vanishes. Well known examples are harmonic functions ($N = \mathbb{R}$), geodesics ($M \subset \mathbb{R}$) and minimal surfaces. Harmonic maps play an important role in many areas of mathematics, see Eells, Lemaire (1978, 1988) for a survey.

The first existence and regularity results for harmonic maps have been derived by Eells, Sampson (1964), considering the parabolic equation $\frac{\partial}{\partial t} f(t,x) = \tau(f)(t,x)$ with given initial map $f(0,.)$ and then letting t go to ∞. An important assumption here is that the target N has nonpositive curvature. Otherwise, the solution may blow up, see e.g. Struwe (1985).

The elliptic approach, based on the fact that classical harmonic maps are critical values of the energy $E(f) := \frac{1}{2}\int_M \|df(x)\|^2 \mathrm{vol}(dx)$, was initiated by Hildebrandt, Kaul, Widman (1975, 1977).

Ishihara (1979) characterized harmonic maps $f : M \to N$ by the fact that for each convex function φ, defined on some open $N_0 \subset N$, the function $\varphi \circ f$, defined on $f^{-1}(N_0) \subset M$, is subharmonic.

In the last decade, for several reasons it was found necessary also to study maps into more general target spaces, e.g. Gromov, Schoen (1992). Ishihara's characterization indicates that any framework for such an extension will require an appropriate notion of subharmonic functions on the domain space and the notion of convex functions on the target space.

Korevaar, Schoen (1993, 1997) and Jost (1994, 1997) independently began to develop a theory of harmonic maps into metric spaces of nonpositive curvature in the sense of Alexandrov (briefly: NPC spaces). These developments are based on the fact that a canonical extension of the energy functional can be defined for maps with values in NPC spaces. In the approach by Korevaar, Schoen, the domain space is still a Riemannian manifold. Generalizations to Lipschitz manifolds and Riemannian polyhedra are due to Gregori (1998) and Eells, Fuglede (2001). In Jost's approach, the domain space is a locally compact metric space with a Dirichlet form on it.

Jost (1997) succeeded to prove Hölder continuity of harmonic maps provided a scale invariant Poincaré inequality holds true on balls of the domain space. For the more specific case of Riemannian domain spaces, Korevaar, Schoen (1993) could prove Lipschitz continuity of harmonic maps.

Our approach admits a more general class of target spaces than the class of NPC spaces. For instance, also Banach spaces and l_p-products of NPC spaces are included. We assume that the target space is a complete metric space (N, d) equipped with a map b which assigns to each probability measure p on N (with bounded support, say) a point $b(p) \in N$, called barycenter or center of mass. The intuitive meaning is that $b(p) = \int_N z\, p(dz)$. Indeed, for Banach spaces this may be used as a definition for b. For NPC spaces we may choose $b(p) := \mathrm{argmin}_{y\in N} \int_N d^2(y,z)\, p(dz)$.

Instead of curvature conditions we require that

$$d(b(q_1), b(q_2)) \le \int_{N\times N} d(y_1, y_2)\, q(dy) \tag{1}$$

for each probability measure q on $N \times N$ with marginals q_1 and q_2. Our basic point of view is that curvature conditions (on domain as well as on target spaces) should be replaced by coupling properties.

Our domain space will be a metric space (M,ρ) with a semigroup of Markov kernels $p_t(x,dy)$ on it. In the classical case, it is given in terms of the heat kernel: $p_t(x,dy) = k_t(x,y)\mathrm{vol}(dy)$. Other examples are derived from SDEs, from elliptic or subelliptic PDEs, from pseudodifferential operators as well as from operators on infinite dimensional spaces. Each Dirichlet form gives rise to such a Markov semigroup. However, we do not require that our semigroups are symmetric whereas in previous approaches symmetry is essential since everything is defined in terms of the energy.

Abstractly spoken, there is some kind of duality: On the domain space M we have (for each $t>0$) a map p_t which assigns to each point x in M a probability measure $p_t(x,.)$ on M. On the target space N we have a map b which assigns to each probability measure q on N a point $b(q)$ in N.

As in the classical approach by Eells, Sampson (1964), we first consider the solution to the parabolic problem. Given a map $f: M\to N$, we define its evolution after time t by

$$P_t^* f := \lim_{\delta_n\to 0} P_{\delta_n}^{\lfloor t/\delta_n\rfloor} f$$

(provided this limit exists for some sequence $(\delta_n)_n$) where

$$P_t f(x) := b\left(p_t(x, f^{-1}(.))\right)$$

denotes the barycenter of the push forward of the probability measure $p_t(x,.)$ under the map f. The intuitive meaning is that $P_t f(x) = \int_M f(y)\, p_t(x,dy)$.

Our main observation is that, under (1), contraction and smoothing properties of the linear semigroup $(p_t)_{t>0}$ carry over to the nonlinear semigroup $(P_t^*)_{t>0}$. For instance, if

$$\mathrm{dil}\, p_t u \le e^{\kappa t}\cdot \mathrm{dil}\, u \tag{2}$$

for all Lipschitz functions $u: M\to\mathbb{R}$ then $P_t^* f$ exists for all Lipschitz maps $f: M\to N$ and

$$\mathrm{dil}\, P_t^* f \le e^{\kappa t}\cdot \mathrm{dil}\, f. \tag{3}$$

More involved assumptions on $(p_t)_{t>0}$ will imply that $P_t^* f$ exists for all bounded maps $f: M\to N$ and

$$\mathrm{dil}\, P_t^* f \le C_t\cdot \mathrm{osc} f.$$

We present many examples, including heat semigroups on manifolds and Alexandrov spaces, convolution semigroups on Lie groups, and Ornstein-Uhlenbeck semigroups on Wiener spaces. Similarly, we prove that the nonlinear operator P_t^* has the same $L_p - L_q$ smoothing properties as the underlying linear operator p_t. Hence, we may use logarithmic Sobolev inequalities and spectral bounds for the generator of the linear semigroup $(p_t)_t$ in order to deduce contraction properties for the nonlinear semigroup $(P_t^*)_t$.

Under weak assumptions, again on $(p_t)_{t>0}$, the maps $P_t^* f$ will converge as $t\to\infty$ to a map h with $P_t^* h = h$ ("invariance"), in particular, with

$$\lim_{t\to 0}\frac{1}{t} d(h, P_t^* h) = 0 \tag{4}$$

("harmonicity"). The solution to the Dirichlet problem on a set $D\subset M$ will be obtained in a similar manner, just replacing the original semigroup by the stopped semigroup $(p_{D,t})_{t>0}$ which preserves boundary data and, in the local case, leads to the same notion of harmonic maps. We prove that under minimal assumptions this nonlinear Dirichlet problem has a unique solution. In addition, under mild restrictions, this solution will be locally Lipschitz continuous in the interior of D and continuous (or even Hölder continuous) at the boundary of D.

In order to see the relation between our notion of harmonic maps and the classical one, let the target N be either a Riemannian manifold or a metric tree or a Banach space. Then (again under some minimal technical assumptions) a map $f : M \to N$ will be harmonic in the sense of (4) if and only if the function $\varphi \circ f$ is subharmonic (w.r.t. $(p_t)_t$) for each Lipschitz continuous convex function $\varphi : N \to \mathbb{R}$.

Our approach also yields a new interpretation of curvature assumptions which are usually required to deduce regularity results for harmonic maps and/or the associated nonlinear heat flow. Let us choose the classical framework where M and N are smooth Riemannian manifolds and $(p_t)_t$ is the heat semigroup (associated with Laplace-Beltrami operator and Brownian motion) on M. In order to deduce the "gradient estimate" (3) (either analytically using Bochner's formula or probabilistically using Bismut's formula) one has to impose lower Ricci curvature bounds on the domain space and upper sectional curvature bounds on the target space. More precisely, one has to require

$$\mathrm{Ric}_M \geq -\kappa, \qquad \mathrm{Sec}_N \leq 0.$$

In our approach, both curvature conditions are replaced by contraction properties in terms of the L_1-Wasserstein distance d^W (see Chapter 2). The condition $\mathrm{Ric}_M \geq -\kappa$ is replaced by

$$d^W\left(p_t(x,.), p_t(y,.)\right) \leq e^{\kappa t} \cdot d(x,y) \tag{5}$$

(for all points $x, y \in M$ and $t > 0$) – which is equivalent to the lower Ricci curvature bound in the Riemannian setting and equivalent to (2) in the general setting.

The condition $\mathrm{Sec}_N \leq 0$ is replaced by

$$d\left(b(q_1), b(q_2)\right) \leq d^W(q_1, q_2) \tag{6}$$

(for all probability measures q_1, q_2 on N) – which is equivalent to the upper sectional curvature bound in the Riemannian setting and equivalent to (1) in the general setting. In particular, there is again a kind of duality: *nonpositive sectional curvature* implies that the distance of two distributions dominates the distance of the respective barycenters whereas *nonnegative Ricci curvature* implies that the distance of two starting points dominates the distance of the distributions at any later time.

We proceed as follows:

In Chapters 1 and 2 we present our basic assumptions on domain and target spaces and illustrate the generality of our framework. We give many examples which are not covered by any of the previous approaches.

Chapter 3 is devoted to the definition of our basic objects: the nonlinear Markov operators P_t and the nonlinear heat operators P_t^*.

In Chapter 4 we derive two fundamental results (Theorem 4.1 and Theorem 4.3) which state that the limit $P_t^* f(x) = \lim_{k\to\infty} P_{\delta_k}^{\lfloor t/\delta_k \rfloor} f(x)$ exists for each Lipschitz continuous map $f : M \to N$ (or even for each bounded measurable f) and defines a Lipschitz continuous map $P_t^* f : M \to N$. Theorem 4.3 gives convergence for *some* sequence $(\delta_k)_k$, Theorem 4.1 yields convergence for *each* sequence. The rather technical proof of the last result is postponed to Chapter 8. It is based on a precise estimate for barycenters ("reverse variance inequality") which may be regarded as a quantitative description of curvature effects.

Chapter 5 deals with the uniform approach and with the L_θ-approach. Various contraction properties will be shown to carry over from the linear semigroup $(p_t)_t$ to the nonlinear semigroup $(P_t^*)_t$.

In Chapter 6 we introduce the concepts of harmonic invariant maps and harmonic maps and we derive existence and uniqueness for the solutions to the nonlinear Dirichlet problem. We also prove that (under appropriate assumptions on the linear semigroup) these solutions are locally Lipschitz continuous in the interior and Hölder continuous at the boundary.

Finally, in Chapter 7 we study harmonic maps with values in manifolds or trees. In particular, we deduce and generalize Ishihara's characterization of harmonic maps in terms of subharmonic and convex functions.

1 The Domain Space

Our domain space will be a measurable space $(M, \mathcal{M})$ **with a given Markov semigroup** $p = (p_t)_{t>0}$ **on it.** That is, M is an arbitrary set, $\mathcal{M}$ is a σ-field on M and $p :]0,\infty[\times M \times \mathcal{M} \to [0,\infty]$ satisfies

- $\forall t > 0, \forall A \subset \mathcal{M} : x \mapsto p_t(x, A)$ is a $\mathcal{M}$-measurable function on M;
- $\forall t > 0, \forall x \in M : A \mapsto p_t(x, A)$ is a probability measure on $(M, \mathcal{M})$;
- $\forall s, t > 0, \forall x \in M, \forall A \in \mathcal{M} :\ p_{s+t}(x, A) = \int_M p_t(y, A) p_s(x, dy)$.

Occasionally, we require $(M, \mathcal{M})$ to be a *Radon measurable space*. All locally compact spaces with countable bases as well as all Polish spaces (= complete separable metric spaces) – equipped with their Borel σ-fields – are Radon measurable spaces.

Example 1.1. Let M be a Riemannian manifold, $\mathcal{M}$ its Borel σ-field, m the Riemannian volume measure and $k :]0,\infty[\times M \times M \to [0,\infty]$ be the minimal heat kernel on M (= fundamental solution of $\frac{1}{2}\Delta - \frac{\partial}{\partial t}$). Then $p_t(x, dy) := k_t(x, y) m(dy)$ defines a Markov semigroup provided M is *stochastically complete*, i.e. provided $p_t(x, M) = 1$ for all $x \in M$ and some (hence all) $t > 0$. The latter is always satisfied if M is connected and complete and if the Ricci curvature of M is bounded from below or, more generally, if $\mathrm{Ric}|_{B(r,x_0)} \geq C(r^2 + 1)$ or, even more generally, if $m(B_r(x_0)) \leq \exp[C(r^2 + 1)]$ (for some $x_0 \in M, C \in \mathbb{R}$ and all $r > 0$), cf. Grigoryan (2000).

Example 1.2. Let $M = \mathbb{R}^{d+1}$ equipped with its Borel σ-field $\mathcal{M}$ and let $k(s, x, t, dy)$ be the transition kernel for the parabolic partial differential equation

$$\frac{\partial}{\partial s} u(x, s) = \sum_{i,j=1}^{d} a_{ij}(x, s) \frac{\partial^2}{\partial x_i \partial x_j} u(x, s) + \sum_{i=1}^{d} b_i(x, s) \frac{\partial}{\partial x_i} u(x, s)$$

on $\mathbb{R}^d$, where a_{ij} and b_i are bounded measurable functions on $\mathbb{R}^{d+1}$ and (a_{ij}) is locally uniformly elliptic, symmetric and continuous. Then $p_t((x, s), A) := \int 1_A((y, s+t)) k(s, x, s+t, dy)$ defines a Markov semigroup on $\mathbb{R}^{d+1}$. If the coefficients a_{ij} and b_i do not depend on time then $p_t(x, B) := k(0, x, t, B)$ defines a Markov semigroup on $\mathbb{R}^d$. See Stroock, Varadhan (1981).

Similar results hold true for hypo- and subelliptic operators (cf. Fefferman, Phong (1983), Jerison, Sanchez-Calle (1986)) as well as for certain pseudodifferential operators, for instance for $(-\Delta)^{\alpha/2}$ with $\alpha < 2$ which is covered by the next result.

Example 1.3. Given a symmetric matrix $a \in \mathbb{R}^{d\times d}$, a vector $b \in \mathbb{R}^d$ and a measure μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} \|y\|^2/(1+\|y\|^2)\mu(dy) < \infty$ there exists a unique convolution semigroup $(q_t)_{t>0}$ of probability measures on $\mathbb{R}^d$ such that $p_t(x,B) := q_t(B-x)$ defines the Markov semigroup for the Lévy operator

$$\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}u(x) + \int \left(u(x+y) - u(x) - \frac{y\cdot\nabla u(x)}{1+\|y\|^2}\right)\mu(dy).$$

See e.g. Ethier, Kurtz (1986), Jacob (1996, 2001), Taira (1991).

Lemma 1.4. *Each quasi-regular conservative Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on a σ-finite measure space $(M,\mathcal{M},m)$ defines a Markov semigroup $(p_t)_{t>0}$ such that $\forall u \in L_2(M)\cap L_\infty(M)$ and for m-a.e. $x \in M$*

$$e^{ta}u(x) = \int_M u(y)p_t(x,dy). \tag{7}$$

Here a denotes the generator of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. See Ma, Röckner (1992).

Standard examples here are Dirichlet forms associated with elliptic differential operators in divergence form (with bounded measurable coefficients) on $\mathbb{R}^d$. Let us mention some non-classical examples of quasi-regular conservative Dirichlet forms:

- Dirichlet form on the Wiener space $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^n)$ and Ornstein-Uhlenbeck semigroup;
- Dirichlet forms on path or loop spaces $\mathcal{C}(\mathbb{R}_+,M)$ or $\mathcal{C}(S^1,M)$, resp., over Riemannian manifolds.

The "quasi-regularity" of the Dirichlet form is not really essential here since in the sequel we only use p_t for $t \in \mathbb{T} := \{k\cdot 2^{-n} : k,n\in\mathbb{N}\}$ and for each conservative Dirichlet form on a Radon space $(M,\mathcal{M})$ there exists a Markov semigroup $(p_t)_{t\in\mathbb{T}}$ satisfying (7).

Lemma 1.5. *Each Markov process $(\Omega,\mathcal{A},\mathbb{P},X_t^x)_{t,x}$ with values in some measurable space $(M,\mathcal{M})$ defines a Markov semigroup on that space by*

$$p_t(x,A) := \mathbb{P}(X_t^x \in A). \tag{8}$$

If $(M,\mathcal{M})$ is a Radon measurable space then vice versa: each Markov semigroup on $(M,\mathcal{M})$ defines via (8) a Markov process (unique up to equivalence). See e.g. Bauer (1996).

One of the main examples for such Markov processes are solutions of stochastic differential equations

$$dX_t^x = x + b(X_t^x)dt + \sigma(X_t^x)dW_t$$

on $\mathbb{R}^d$ where $(W_t)_t$ denotes Brownian motion and $b:\mathbb{R}^d\to\mathbb{R}$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^d$ are locally Lipschitz and bounded. Other remarkable examples are

- Super-Brownian motion on the space of measures on $\mathbb{R}^n$;
- Fleming-Viot processes on the space of probability measures on $\mathbb{R}^n$;
- Interacting particle systems as processes on the configuration space over $\mathbb{R}^n$.

In all the above mentioned examples, the Markov processes can be chosen to be *right Markov processes* which means that they have some additional minimal regularity properties. All Lévy, Feller, Hunt, and standard processes are right processes.

Definition 1.6. *Let $(\Omega, \mathcal{A}, \mathbb{P}, X_t^x)_{t,x}$ be a right Markov process associated with a Markov semigroup $(p_t)_{t>0}$ on a Radon measurable space $(M, \mathcal{M})$. Then for each measurable subset $D \subset M$ the stopped semigroup $(p_{D,t})_{t>0}$ is the Markov semigroup on $(M, \mathcal{M})$ defined by*

$$p_{D,t}(x, A) := \mathbb{P}(X^x_{t\wedge\tau(D,x)} \in A)$$

where $\tau(D, x) := \inf\{t \geq 0 : X_t^x \notin D\}$ denotes the first exit time of D.

Definition 1.7. Given a Markov semigroup $(p_t)_{t>0}$ on a measurable space $(M, \mathcal{M})$ we define for each $t > 0$ the *terminal coupling operator* p_t^∇ acting on symmetric functions $\rho : M \times M \to \mathbb{R}_+$ by

$$p_t^\nabla \rho(x_1, x_2) := \sup_u |p_t u(x_1) - p_t u(x_2)|$$

where the supremum is over all bounded measurable $u : M \to \mathbb{R}$ satisfying $|u(y_1) - u(y_2)| \leq \rho(y_1, y_2)$ for all $y_1, y_2 \in M$. The *coupling semigroup* $(p_t^\diamond)_{t>0}$ acting on symmetric functions $\rho : M \times M \to \mathbb{R}_+$ is defined by

$$p_t^\diamond \rho(x_1, x_2) := \sup \left\{ p_{t_n}^\nabla \circ \ldots \circ p_{t_1}^\nabla \rho(x_1, x_2) : n \in \mathbb{N}, t_i > 0, \sum_{i=1}^n t_i = t \right\}.$$

Remark 1.8. For each Markov semigroup $(p_t)_{t>0}$ on a measurable space $(M, \mathcal{M})$

$$p_t^\nabla \rho(x_1, x_2) \leq \inf \mathbb{E}\rho(Z_1, Z_2) \tag{9}$$

where the infimum is over all probability spaces $(\Omega, \mathcal{A}, \mathbb{P})$ and all random variables $Z_i : \Omega \to N$ with distribution $\mathbb{P}(Z_i \in .) = p_t(x_i, .)$ (for $i = 1, 2$). Similarly,

$$p_t^\diamond \rho(x_1, x_2) \leq \inf \mathbb{E}\rho(X_1^{x_1,x_2}(t), X_2^{x_1,x_2}(t)) \tag{10}$$

where the infimum is over all Markov processes $(\Omega, \mathcal{A}, \mathbb{P}, (X_1^{x_1,x_2}(t), X_2^{x_1,x_2}(t))_{t,x_1,x_2}$ on $M \times M$ for which the marginal processes $(\Omega, \mathcal{A}, \mathbb{P}, (X_1^{x_1,x_2}(t))_{t,x_1}$ and $(\Omega, \mathcal{A}, \mathbb{P}, (X_2^{x_1,x_2}(t))_{t,x_2}$ have transition semigroup $(p_t)_{t>0}$.

Moreover, under weak regularity assumptions the above inequalities are indeed equalities. For instance, if ρ is a complete separable metric on M and if $\mathcal{M}$ is its Borel σ-field then (9) is an equality. In general, using the notation from the next Chapter the RHS of (9) equals $\rho^W(p_t(x_1, .), p_t(x_2, .))$. Cf. Rachev, Rüschendorf (1998), Kendall (1990).

2 The Target Space

Our target space will be a complete metric space (N, d) with a given barycenter contraction b on it.

We denote by $\mathcal{N}$ the Borel σ-field of N, and for each $\theta > 0$, by $\mathcal{P}^\theta(N)$ the set of all probability measures p on $(N, \mathcal{N})$ with separable support and with $\int_N d^\theta(z, x)p(dx) < \infty$ for some/all $z \in N$. Given two measures $p, q \in \mathcal{P}^1(N)$, a measure $\mu \in \mathcal{P}^1(N \times N)$ is called *coupling* of p and q iff

$$\mu(A \times N) = p(A), \quad \mu(N \times A) = q(A) \qquad (\forall A \in \mathcal{N}).$$

The L_1-*Wasserstein distance* or *Kantorovich-Rubinstein distance* of $p, q \in \mathcal{P}^1(N)$ is defined as

$$d^W(p, q) = \inf \left\{ \int_{N^2} d(x_1, x_2)\mu(dx) : \mu \in \mathcal{P}^1(N^2) \text{ is a coupling of } p \text{ and } q \right\}.$$

Definition 2.1. A *barycenter contraction* is a map $b : \mathcal{P}^1(N) \to N$ such that

- $b(\delta_x) = x$ for all $x \in N$;
- $d(b(p), b(q)) \le d^W(p,q)$ for all $p, q \in \mathcal{P}^1(N)$.

Remark 2.2. If there exists a barycenter contraction on (N,d) then (N,d) is a geodesic space: For each pair of points $x_0, x_1 \in N$ we can define one geodesic $t \mapsto x_t$ connecting x_0 and x_1 by $x_t := b((1-t)\delta_{x_0} + t\delta_{x_1})$.

Given any four points $x_0, x_1, y_0, y_1 \in N$, the function $t \mapsto d(x_t, y_t)$ is convex. Indeed,

$$d(x_t, y_t) \le d^W((1-t)\delta_{x_0} + t\delta_{x_1}, (1-t)\delta_{y_0} + t\delta_{y_1}) \le (1-t)d(x_0,y_0) + td(x_1,y_1)$$

since $(1-t)\delta_{(x_0,y_0)} + t\delta_{(x_1,y_1)}$ is a coupling of $(1-t)\delta_{x_0} + t\delta_{x_1}$ and $(1-t)\delta_{y_0} + t\delta_{y_1}$.

In particular, the geodesic $t \mapsto x_t$ depends continuously on x_0 and x_1. However, it is not necessarily the only geodesic connecting x_0 and x_1.

If geodesics in N are unique then the existence of a barycenter contraction implies that $d : N \times N \to \mathbb{R}$ is convex. Thus N has globally "nonpositive curvature" in the sense of Busemann.

Example 2.3. Let (N,d) be a complete metric space with globally "nonpositive curvature" in the sense of A.D. Alexandrov. Then for each $p \in \mathcal{P}^2(N)$ there exists a unique $b(p) \in N$ which minimizes the uniformly convex function

$$z \mapsto \int_N d^2(z,x)p(dx)$$

on N. The map $b : \mathcal{P}^2(N) \to N$ extends to a barycenter contraction $\mathcal{P}^1(N) \to N$. See Sturm (2001).
Equivalently, $b(p)$ can be defined via the law of large numbers as the unique accumulation point of the sequence

$$\frac{1}{n}\sum_{i=1}^{n} X_i(w)$$

for a.e. ω where $(X_i)_i$ is a sequence of independent random variables with distribution p. The point $\frac{1}{n}\sum_{i=1}^{n} X_i(w)$ is defined by induction on n as the point $\gamma_{1/n}$ on the geodesic from $\gamma_0 := \frac{1}{n-1}\sum_{i=1}^{n-1} X_i(w)$ to $\gamma_1 := X_n(w)$. See Sturm (2002). Examples of spaces with globally nonpositive curvature in the sense of A.D. Alexandrov are

- complete, simply connected Riemannian manifolds with nonpositive sectional curvature;
- trees and, more generally, Euclidean Bruhat-Tits buildings;
- Hilbert spaces;
- L_2-spaces of maps into such spaces;
- Finite or infinite (weighted) products of such spaces;
- Gromov-Hausdorff limits of such spaces.

See e.g. Ballmann (1995), Bridson, Haefliger (1999), Burago, Burago, Ivanov (2001), Eells, Fuglede (2001), Gromov (1999), Jost (1994, 1997a), Korevaar, Schoen (1993, 1997).

Example 2.4. Let N be a complete, simply connected Riemannian manifold and let d be a Riemannian distance. *Then (N,d) admits a barycenter contraction b if and only if N has nonpositive sectional curvature.*
Indeed, if (N,d) admits a barycenter contraction then so does (N_0,d) for each closed convex $N_0 \subset N$. Hence, geodesics in N_0 are unique and thus $t \mapsto d(\gamma_t, \zeta_t)$ is convex for any pair of geodesics γ and ζ in N_0. This implies that N has nonpositive curvature (Jost (1997a)).
Conversely, if N has nonpositive curvature then it admits a barycenter contraction by the previous Example 2.3.

Example 2.5. Let (N,d) be a locally compact separable complete metric space with *negative curvature in the sense of Busemann.* Then Es-Sahib, Heinich (1999) have constructed a barycenter contraction. For Riemannian manifolds, this is different from those in Examples 2.3 and 2.4, and also for trees, it is different from that in Example 2.3.

Example 2.6. Let $(N, \|.\|)$ be a (real or complex) Banach space and put $d(x,y) := \|x-y\|$. Then $\mathcal{P}^1(N)$ is the set of Radon measures p on N satisfying $\int_N \|x\|\, p(dx) < \infty$. For each $p \in \mathcal{P}^1(N)$, the identity $x \mapsto x$ on N is *Bochner integrable* and

$$b(p) := \int_N x\, p(dx)$$

defines a barycenter contraction on (N,d). Cf. Ledoux, Talagrand (1991), for instance.

Lemma 2.7. *Let I be a countable set and for each $i \in I$, let (N_i, d_i) be a complete metric space with barycenter contraction b_i and "base" point $o_i \in N_i$. Given $\theta \in [1,\infty]$, define a complete metric space (N,d) with base point $o = (o_i)_{i\in I}$ by*

$$N := \left\{ x = (x_i)_{i\in I} \in \bigotimes_{i\in I} N_i : \ d(x,o) < \infty \right\}, \quad d(x,y) := \left[\sum_{i\in I} d_i^\theta(x_i,y_i) \right]^{\frac{1}{\theta}}$$

provided $\theta < \infty$ or by $d(x,y) = \sup_{i\in I} d_i(x_i,y_i)$ if $\theta = \infty$. One can define a barycenter contraction b on $\mathcal{P}^1(N)$ by

$$b(p) := (b_i(p_i))_{i\in I}$$

where $p_i \in \mathcal{P}^1(N_i)$ with $p_i : A \mapsto p(\{x = (x_j)_{j\in I} \in N : x_i \in A\})$ denotes the projection of $p \in \mathcal{P}^1(N)$ onto the i-th factor of N.

Proof. Let π_i denote the projection $N \to N_i$. For $\theta = \infty$

$$\begin{aligned} d(b(p), b(q)) &= \sup_{i\in I} d_i(b_i(p_i), b_i(q_i)) \\ &\le \sup_{i\in I} \inf \left\{ \int_{N_i\times N_i} d_i(x_i,y_i) d\mu_i((x_i,y_i)) : \mu_i \text{ coupling of } \pi_i(p) \text{ and } \pi_i(q) \right\} \\ &\le \sup_{i\in I} \inf \left\{ \int_{N\times N} d_i(\pi_i(x), \pi_i(y)) d\mu((x,y)) : \mu \text{ coupling of } p \text{ and } q \right\} \\ &\le \inf \left\{ \int_{N\times N} d(x,y) d\mu((x,y)) : \mu \text{ coupling of } p \text{ and } q \right\} = d^W(p,q) \end{aligned}$$

and similarly for $\theta < \infty$

$$\begin{aligned}
d(b(p), b(q)) &\leq \left[\sum_{i \in I} \inf_\mu \left[\int_{N \times N} d_i(\pi_i(x), \pi_i(y)) d\mu((x,y)) \right]^\theta \right]^{\frac{1}{\theta}} \\
&\leq \inf_\mu \left[\sum_{i \in I} \left[\int_{N \times N} d_i(\pi_i(x), \pi_i(y)) d\mu((x,y)) \right]^\theta \right]^{\frac{1}{\theta}} \\
&\leq \inf_\mu \int_{N \times N} \left[\sum_{i \in I} d_i^\theta(\pi_i(x), \pi_i(y)) \right]^{\frac{1}{\theta}} d\mu((x,y)) \;=\; d^W(p,q)
\end{aligned}$$

where $\inf_\mu$ always denotes the infimum over all couplings $\mu \in \mathcal{P}^1(N \times N)$ of p and q. For the last inequality, note that by Minkowski's inequality

$$\left[\sum_{i \in I} \left[\sum_{j \in J} |a(i,j)| \right]^\theta \right]^{1/\theta} \leq \sum_{j \in J} \left[\sum_{i \in I} |a(i,j)|^\theta \right]^{1/\theta}$$

for all finite sets J and all sequences $a(i,j)$ which extends (by the usual measure theoretic arguments) to

$$\left[\sum_{i \in I} \left[\int_X |a(i,\xi)\, \mu(d\xi)| \right]^\theta \right]^{1/\theta} \leq \int_X \left[\sum_{i \in I} |a(i,\xi)|^\theta \right]^{1/\theta} \mu(d\xi)$$

for all probability measures μ and all measurable functions $a(i,.)$ on $X = M \times M$. $\square$

For instance, this applies to $N = \mathbb{R}^n, n \geq 2$ with the usual notion of barycenter but with "unusual" metric $d(x,y) = \sup\{|x_i - y_i| : \; i = 1,..,n\}$. In this case, geodesics are not unique, e.g. each curve $t \mapsto (t, \varphi_2(t), ..., \varphi_n(t))$ with $\varphi \in \mathcal{C}^1(\mathbb{R}), \varphi_i(0) = \varphi_i(1) = 0$ and $|\varphi_i'| \leq 1$ is a geodesic connecting $(0,0,...,0)$ and $(1,0,...,0)$.

Each barycenter map b on a complete metric space (N,d) gives rise to a whole family of barycenter maps b_n, $n \in \mathbb{N}$ (which in general do not coincide with b, see Example below).

Proposition 2.8. *Let (N,d,b) be a barycentric metric space and $\Phi : N \times N \to N$ be the "midpoint map" induced by b, i.e. $\Phi(x,y) = b(\frac{1}{2}\delta_x + \frac{1}{2}\delta_y)$. Define a map $\Xi : \mathcal{P}^1(N) \to \mathcal{P}^1(N)$ by*

$$\Xi(q) := \Phi_*(q \otimes q).$$

Then Ξ is a contraction with respect to d^W. Thus for each $n \in \mathbb{N}$

$$b_n(q) := b(\Xi^n(q))$$

defines a barycenter map $b_n : \mathcal{P}^1(N) \to N$.

Proof. It suffices to prove that Ξ is a contraction on $(\mathcal{P}^1(N), d^W)$, i.e. that $d^W(\Xi(p), \Xi(q)) \leq d^W(p,q)$ for each pair $p,q \in \mathcal{P}^1(N)$. Let $\mu \in \mathcal{P}(N^2)$ be an optimal coupling of p and q. Without restriction, we may assume $p = \frac{1}{k}\sum_{i=1}^k \delta_{x_i}$, $q = \frac{1}{k}\sum_{i=1}^k \delta_{y_i}$ and $\mu = \frac{1}{k}\sum_{i=1}^k \delta_{(x_i,y_i)}$. Let $x_{ij} := \Phi(x_i, x_j)$ and $y_{ij} := \Phi(y_i, y_j)$. Then $\Xi(p) = \frac{1}{k^2}\sum_{i,j=1}^k \delta_{x_{ij}}$ and $\Xi(q) = \frac{1}{k^2}\sum_{i,j=1}^k \delta_{y_{ij}}$. Define a coupling μ' of $\Xi(p)$ and $\Xi(q)$ by $\mu' = \frac{1}{k^2}\sum_{i,j=1}^k \delta_{(x_{ij},y_{ij})}$. Note that $d(x_{ij}, y_{ij}) \leq d^W(\frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_j}, \frac{1}{2}\delta_{y_i} + \frac{1}{2}\delta_{y_j}) \leq \frac{1}{2}d(x_i,y_i) + \frac{1}{2}d(x_j,y_j)$. Hence, $d^W(\Xi(p), \Xi(q)) \leq \frac{1}{k^2}\sum_{i,j=1}^k d(x_{ij}, y_{ij}) \leq \frac{1}{2k^2}\sum_{i,j=1}^k [d(x_i,y_i) + d(x_j,y_j)] = \frac{1}{k}\sum_{i=1}^k d(x_i,y_i) = d^W(p,q)$. $\square$

Example 2.9. Define the *tripod* by gluing together 3 copies of $\mathbb{R}_+$ at their origins, i.e.

$$N = \{(i,r) : i \in \{1,2,3\}, r \in \mathbb{R}_+\}/\sim \quad \text{where } (i,r) \sim (j,s) :\Leftrightarrow r = s = 0.$$

It can be realized as the subset $\{r \cdot \exp(\frac{l}{3}2\pi i) \in \mathbb{C} : r \in \mathbb{R}_+, l \in \{1,2,3\}\}$ of the complex plane, however, equipped with the (non-Euclidean!) intrinsic metric

$$d((i,r),(j,s)) = \begin{cases} |r-s|, & \text{if } i = j \\ |r| + |s|, & \text{else.} \end{cases}$$

Then (N,d) is a complete metric space of globally nonpositive curvature and according to Example 2.3 there exists a canonical barycenter map b. Derive from that the barycenter map $b_1 = b(\Xi(.))$ as above. Then the maps b and b_1 do not coincide. Indeed, choose $q = \frac{1}{2}\delta_{(1,1)} + \frac{1}{4}\delta_{(2,1)} + \frac{1}{4}\delta_{(3,1)}$. Then $\Xi(q) = \frac{1}{4}\delta_{(1,1)} + \frac{1}{16}\delta_{(2,1)} + \frac{1}{16}\delta_{(3,1)} + \frac{5}{8}\delta_o$. Hence, $b(q) = (1,0)$ and $b_1(q) = b(\Xi(q)) = (1, \frac{1}{8})$.

3 The Nonlinear Heat Semigroup

Let $(M,\mathcal{M})$, $p = (p_t)_{t>0}$ and (N,d,b) be as in Chapters 1, 2 and let $\mathcal{L}(M,N,p)$ denote the set of all measurable maps $f : M \to N$ with separable ranges and with

$$\eta_t f(x) := \int_M d(f(x), f(y)) p_t(x,dy) < \infty$$

for all $t > 0$ and all $x \in M$. For each such f, t and x, the probability measure $p_t(x, f^{-1}(\bm{.}))$ lies in $\mathcal{P}^1(N)$ and thus

$$P_t f(x) := b(p_t(x, f^{-1}(\bm{.})))$$

is well-defined.

Lemma 3.1. *For all $f, g \in \mathcal{L}(M,N,p)$, all $s,t > 0$, and all $x,y \in M$*

(i) $d(P_t f(x), P_t g(x)) \le \int d(f(y), g(y)) p_t(x,dy)$;

(ii) $d(P_t f(x), f(x)) \le \eta_t f(x)$;

(iii) $\eta_s(P_t f)(x) \le \eta_{s+t} f(x) + \eta_t f(x)$;

(iv) $d(P_t f(x), P_t f(y)) \le p_t^\nabla d_f(x,y)$ with p_t^∇ from Definition 1.7 and d_f denoting the function $(x,y) \mapsto d(f(x), f(y))$ on $M \times M$;

(v) $P_t f \in \mathcal{L}(M,N,p)$.

The map $P_t : \mathcal{L}(M,N,p) \to \mathcal{L}(M,N,p)$ is called nonlinear Markov operator associated with the kernel p_t.

Proof. (i) By the defining property of barycenter contractions

$$d(P_t f(x), P_t g(x)) \le d^W\left(p_t(x, f^{-1}(\bm{.})), p_t(x, g^{-1}(\bm{.}))\right) \le \int d(f(y), g(y)) p_t(x,dy).$$

(ii) Choose $g(\centerdot) \equiv f(x)$ in (i).
(iii) Using (i) we obtain

$$\begin{aligned}
\eta_s(P_t f)(x) &= \int d(P_t f(x), P_t f(y)) p_s(x, dy) \\
&\leq d(f(x), P_t f(x)) + \int d(f(x), P_t f(y)) p_s(x, dy) \\
&\leq \int d(f(x), f(y)) p_t(x, dy) + \int \int d(f(x), f(z)) p_t(y, dz) p_s(x, dy) \\
&= \eta_t f(x) + \eta_{s+t} f(x).
\end{aligned}$$

(iv) Again by the defining property of barycenter contractions and according to the Kantorovich-Rubinstein duality (see e.g. Rachev, Rüschendorf (1998))

$$\begin{aligned}
& d(P_t f(x), P_t f(y)) \\
&\leq d^W(p_t(x, f^{-1}(\centerdot)), p_t(y, f^{-1}(\centerdot))) \\
&\leq \sup \Big\{ \int_N u(f(z))\, p_t(x, dz) - \int_N u(f(z))\, p_t(y, dz) : \\
&\qquad\qquad u : N \to \mathbb{R} \text{ bdd. meas. with } u(z) - u(z') \leq d(z, z') \ (\forall z, z' \in N) \Big\} \\
&\leq p_t^{\nabla} d_f(x, y).
\end{aligned}$$

(v) It remains to prove that $x \mapsto P_t f(x)$ is measurable and has separable range. This follows as in Sturm(2001), Lemma 6.4. □

For the sequel, fix once for all a subsequence $(\delta_n)_{n\in\mathbb{N}}$ of $(2^{-n})_{n\in\mathbb{N}}$ and put $\mathbb{T} = \{k \cdot 2^{-n} : k, n \in \mathbb{N}\}$. Let $\mathcal{L}^*(M, N, p)$ denote the set of all $f \in \mathcal{L}(M, N, p)$ for which

$$P_t^* f(x) := \lim_{n\to\infty} (P_{\delta_n})^{t/\delta_n} f(x)$$

exists for all $t \in \mathbb{T}$ and all $x \in M$. Here $(P_\delta)^k$ denotes the k-th iteration of the nonlinear Markov operator P_δ.

Note that if $N = \mathbb{R}$ (equipped with the usual d and b) then $\mathcal{L}^*(M, \mathbb{R}, p) = \mathcal{L}(M, \mathbb{R}, p)$ is the set of all measurable $f : M \to \mathbb{R}$ with $\int |f(y)| p_t(x, dy) < \infty$ $(\forall t > 0, x \in M)$, and

$$P_t^* f(x) = P_t f(x) = \int_M f(y) p_t(x, dy).$$

Lemma 3.2. *For all $f, g \in \mathcal{L}^*(M, N, p)$, all $s, t \in \mathbb{T}$ and all $x, y \in M$*

(i) $d(P_t^* f(x), P_t^* g(x)) \leq \int d(f(y), g(y)) p_t(x, dy)$;

(ii) $d(P_t^* f(x), f(x)) \leq \eta_t f(x)$ and $d(P_t^* f(x), P_{t+s}^* f(x)) \leq p_t(\eta_s f)(x)$;

(iii) $\eta_s(P_t^* f)(x) \leq \eta_{s+t} f(x) + \eta_t f(x)$;

(iv) $d(P_t f(x), P_t f(y)) \leq p_t^{\diamond} d_f(x, y)$ with $p_t^{\diamond}$ from Definition 1.7 and d_f denoting the function $(x, y) \mapsto d(f(x), f(y))$ on $M \times M$;

(v) $P_t^* f \in \mathcal{L}^*(M, N, p)$ and $P_s^*(P_t^* f)(x) = P_{s+t}^* f(x)$.

The operator P_t^* on $\mathcal{L}^*(M,N,p)$ is called nonlinear heat operator associated with the "linear heat semigroup" $(p_t)_{t>0}$. The semigroup $(P_t^*)_{t\in\mathbb{T}}$ of operators on $\mathcal{L}^*(M,N,p)$ is called nonlinear heat semigroup.

Proof. (i) Using Lemma 3.1 (i) we conclude $\forall \delta>0, \forall k\in\mathbb{N}$

$$\begin{aligned} d(P_\delta^k f(x), P_\delta^k g(x)) &\le \int d(P_\delta^{k-1} f(x_1), P_\delta^{k-1} g(x_1)) p_\delta(x,dx_1) \\ &\le \int d(P_\delta^{k-2} f(x_2), P_\delta^{k-2} g(x_2)) p_{2\delta}(x,dx_2) \\ &\le \ldots \\ &\le \int d(f(x_k), g(x_k)) P_{k\delta}(x,dx_k). \end{aligned}$$

Hence,

$$d(P_t^* f(x), P_t^* g(x)) = \lim_{n\to\infty} d(P_{\delta_n}^{t/\delta_n} f(x), P_{\delta_n}^{t/\delta_n} g(x)) \le \int d(f(y),g(y)) p_t(x,dy).$$

(ii) By (i)

$$\begin{aligned} d(P_t^* f(x), P_{t+s}^* f(x)) &\le \int d(f(y), P_s^* f(y)) p_t(x,dy) \\ &\le \int \eta_s f(y) p_t(x,dy) = p_t \eta_s f(x). \end{aligned}$$

(iii) follow from (i) choosing $g \equiv f(x)$, cf. Lemma 3.1.
(iv) From Lemma 3.1 (iv) we deduce that $\forall \delta>0, \forall k\in\mathbb{N}$

$$d_{P_\delta^k f}(x,y) \le p_\delta^\triangledown \left(d_{P_\delta^{k-1} f} \right)(x,y) \le \ldots \le (p_\delta^\triangledown)^k (d_f)(x,y) \le p_{k\delta}^\diamond (d_f)(x,y)$$

and thus

$$d(P_t^* f(x), P_t^* f(y)) = \lim_{n\to\infty} d(P_{\delta_n}^{t/\delta_n} f(x), P_{\delta_n}^{t/\delta_n} f(y)) \le p_t^\diamond d_f(x,y).$$

(v) $f_t := P_t^* f$ is the limit of measurable maps with separable range and thus is measurable and has separable range. Together with (i), this implies $f_t \in \mathcal{L}(M,N,p)$. The fact that $f_t \in \mathcal{L}^*(M,N,p)$ and the semigroup property follow from the existence of $P_{s+t}^* f$ and from

$$\begin{aligned} d(\lim_{n\to\infty} P_{\delta_n}^{s/\delta_n} f_t(x), P_{s+t}^* f(x)) &= \lim_{n\to\infty} d(P_{\delta_n}^{s/\delta_n} f_t(x), P_{\delta_n}^{(s+t)/\delta_n} f(x)) \\ &\le \lim_{n\to\infty} \int d(f_t(y), P_{\delta_n}^{t/\delta_n} f(y)) p_s(x,dy) = 0. \end{aligned}$$

The last equality is due to Lebesgue's dominated convergence theorem since $\lim_{n\to\infty} d(f_t(y), P_{\delta_n}^{t/\delta_n} f(y)) = 0$ for each $y\in M$ and, moreover, for all (sufficiently large) $n\in\mathbb{N}$ (and any $z\in N$)

$$d(f_t(y), P_{\delta_n}^{t/\delta_n} f(y)) \le 2\eta_t(z,f)(y)$$

with $\eta_t(z,f)(y) := \int d(z,f(u)) p_t(y,du)$ and $\int \eta_t(z,f)(y) p_s(x,dy) = \eta_{s+t}(z,f)(x) < \infty$. □

For previous approaches to harmonic maps based on iterated barycenters, see Kendall (1990), Picard (1994) and Jost (1994). For other probabilistic approaches, see e.g. Arnaudon (1994), Kendall (1998) and Thalmaier (1996, 1996a). For analytic constructions of a nonlinear heat flow as a gradient flow for generalized harmonic maps, see Jost (1998) and Mayer (1998).

4 Convergence and Lipschitz Continuity

Let $(M, \mathcal{M})$, $p = (p_t)_{t>0}$ and (N, d, b) be as before. In addition, throughout this Chapter we fix a nonnegative symmetric function ρ on $M \times M$. (Typically, ρ will be a metric on M. But $\mathcal{M}$ will not necessarily be the Borel σ-field of ρ.) For $f, g : M \to N$ put $d_\infty(f, g) := \sup_{x \in M} d(f(x), g(x))$,

$$\mathrm{dil}_\rho f := \sup_{x,y \in M} \frac{d(f(x), f(y))}{\rho(x, y)}$$

(with $\frac{0}{0} := 0$) and let $\mathrm{Lip}_\rho(M, N)$ denote the set of measurable $f : M \to N$ with separable range and $\mathrm{dil}_\rho f < \infty$. Moreover, let $L_\infty(M, N)$ denote the set of bounded measurable $f : M \to N$ with separable range. Finally, let henceforth $\lfloor s \rfloor$ denote the integer part of $s \in \mathbb{R}$.

Theorem 4.1. *Assume that (N, d) has globally nonpositive and lower bounded curvature (in the sense of Alexandrov). Moreover, assume $\exists C, \beta > 0$ and $\forall t > 0 : \exists c_t$ such that $\sup_{s \le t} c_s < \infty$ and $\forall x, y \in M$:*

$$\begin{aligned} p_t^\diamond \rho(x, y) &\le c_t \cdot \rho(x, y) &(11)\\ \int \rho^4(x, z) p_t(x, dz) &\le C \cdot t^{1+\beta} &(12) \end{aligned}$$

Then $\mathrm{Lip}_\rho(M, N) \subset \mathcal{L}^*(M, N, p)$ and $(P_t^*)_{t \in \mathbb{R}_+}$ is a strongly continuous semigroup on $\mathrm{Lip}_\rho(M, N)$. More precisely, for all $x \in M, t \in \mathbb{R}_+$ and $f \in \mathrm{Lip}_\rho(M, N)$

$$P_t^* f(x) = \lim_{s \to 0} P_s^{\lfloor t/s \rfloor} f(x)$$

exists and the limit is continuous in each variable:

$$\begin{aligned} \mathrm{dil}_\rho P_t^* f &\le c_t \cdot \mathrm{dil}_\rho f &(13)\\ d_\infty(P_t^* f, P_t^* g) &\le d_\infty(f, g) &(14)\\ d_\infty(P_s^* f, P_t^* f) &\le C^{1/4} \cdot \mathrm{dil}_\rho f \cdot |t - s|^{\frac{1+\beta}{4}}. &(15) \end{aligned}$$

Remark 4.2. (i) If $c_t = e^{\kappa t}$ for some $\kappa \in \mathbb{R}$ and $N = \mathbb{R}$ then condition (11) is already *necessary* for (13). Indeed, for any Markov semigroup $(p_t)_t$ on a metric space (M, ρ) and any $\kappa \in \mathbb{R}$ the following are equivalent:

- $\mathrm{dil}_\rho p_t u \le e^{\kappa t} \cdot \mathrm{dil}_\rho u \qquad (\forall t, \forall u \in \mathrm{Lip}_\rho(M, \mathbb{R}))$
- $p_t^\diamond \rho(x, y) \le e^{\kappa t} \rho(x, y). \qquad (\forall t)$

(ii) Condition (12) is well-known from the theorem of Kolmogorov and Chentsov. It implies (under minimal regularity assumptions) that the Markov process associated with $(p_t)_t$ has continuous paths. Hence, it excludes jump or jump-diffusion processes. However, for diffusions it is a very weak assumption. E.g. for solutions of SDEs with bounded measurable coefficients on $\mathbb{R}^d$ or for the Markov semigroups from Example 1.2, $\int |x - y|^4 p_t(x, dy) \le C \cdot t^2$ for all $x, y \in \mathbb{R}^d$ and all t.

(iii) The assumption on the lower bounded curvature of (N, d) can be weakened in order to include also "spaces with a reverse variance inequality of some order > 2", e.g. the result of gluing together two copies of the set $\{z = (x, t) \in \mathbb{R}^k : t \le \psi(x)\}$ along their boundary $\{z = (x, t) \in \mathbb{R}^k : t = \psi(x)\}$ where $\psi : \mathbb{R}^{k-1} \to \mathbb{R}$ is any smooth convex function.

The proof of Theorem 4.1 and several extensions of it will be given in Chapter 8. For typical examples satisfying (11), see Examples 4.5 - 4.9 below.

The main point in Theorem 4.1 is that it yields convergence independent of the choice of the sequence $(\delta_n)_n$. Our next result will give convergence for suitable choices of sequences $(\delta_n)_n$. Here the advantages will be:
- it does not require any kind of lower curvature bound for (N, d);
- it applies also to jump processes and to nonlocal equations;
- it also yields smoothing from $L_\infty(M, N)$ to $\mathrm{Lip}_\rho(M, N)$.

In order to formulate the latter, let ρ_0 be another nonnegative symmetric function on $M \times M$ (besides ρ) and define dil_{ρ_0} and $\mathrm{Lip}_{\rho_0}(M, N)$ in an analogous way.

Theorem 4.3. *Assume that $(M, \mathcal{M})$ is a Radon measurable space, (M, ρ) is separable and (N, d) is locally compact. Moreover, assume that $\forall t \in \mathbb{T} : \exists C_t : \forall x, y \in M$:*

$$p_t^\diamond \rho_0(x, y) \le C_t \cdot \rho(x, y) \tag{16}$$

$$\int \rho_0(x, z) p_t(x, dz) < \infty. \tag{17}$$

Then for each sequence $(s_k)_k \subset (2^{-k})_k$ there exists a subsequence $(\delta_k)_k = (s_{n_k})_k$ such that $\mathrm{Lip}_{\rho_0}(M, N) \subset \mathcal{L}^*(M, N, p)$ *and for each $t \in \mathbb{T}$*

$$P_t^* : \mathrm{Lip}_{\rho_0}(M, N) \to \mathrm{Lip}_\rho(M, N).$$

More precisely, $\forall f \in \mathrm{Lip}_{\rho_0}(M, N), \forall x \in M, \forall t \in \mathbb{T}$ the limit

$$P_t^* f(x) = \lim_{k \to \infty} P_{\delta_k}^{t/\delta_k} f(x)$$

exists and

$$\mathrm{dil}_\rho P_t^* f \le C_t \cdot \mathrm{dil}_{\rho_0} f. \tag{18}$$

Remark 4.4. (i) The most important choices for ρ_0 are either $\rho_0 \equiv \rho$ or $\rho_0 \equiv 1$. In the latter case, $\mathrm{dil}_{\rho_0} f = \mathrm{osc} f := \sup_{x, y \in M} d(f(x), f(y))$ and thus $\mathrm{Lip}_{\rho_0}(M, N) = L_\infty(M, N)$. Theorem 4.3 then proves existence of the limit $P_t^* f$ for all $f \in L_\infty(M, N)$ and smoothing $P_t^* f : L_\infty(M, N) \to \mathrm{Lip}_\rho(M, N)$.

(ii) If in addition to the assumptions of the previous Theorem $\gamma_s(x) := \int \rho(x, z) p_s(x, dz) \to 0$ for $s \to 0$ then the limit

$$P_t^* f(x) = \lim_{k \to \infty} P_{\delta_k}^{\lfloor t/\delta_k \rfloor} f(x)$$

exists for all $t \in \mathbb{R}_+$. Moreover, for $s \to 0$

$$d(P_t^* f(x), P_{t+s}^* f(x)) \le C_t \cdot \mathrm{dil}_{\rho_0} f \cdot \gamma_s(x) \to 0. \tag{19}$$

(iii) Assume that (16) only holds true for all x, y in a ρ-open set $M_1 \subset M$. Then the limit $P_t^* f(x) = \lim_{k \to \infty} P_{\delta_k}^{t/\delta_k} f(x)$ exists $\forall f \in \mathrm{Lip}_{\rho_0}(M, N), \forall x \in M_1, \forall t \in \mathbb{T}$ and is ρ-Lipschitz continuous on M_1.

Proof. (a) Given $t \in \mathbb{T}$ and $x \in M$, define a metric d_1 by $d_1(f, g) := \int_M d(f(z), g(z)) p_t(x, dz)$ and let $L_1((M, \mathcal{M}, p_t(x, .)), (N, d))$ denote the complete metric space of all measurable maps $f : M \to N$ with separable range and finite d_1-distance from constant maps. For such f, consider $z_s = z_s(t,x,f) := P_s^{t/s} f(x)$ for (sufficiently small) $s \in \{2^{-k} : k \in \mathbb{N}\}$. Due to Lemma (3.2) this is well-defined and contained in a closed ball around $f(x)$:

$$d(f(x), z_s) \le \int d(f(x), f(y)) p_t(x, dy) = d_1(f(x), f) < \infty$$

where $f(x)$ also denotes the constant map $y \mapsto f(x)$.

Due to the local compactness of (N,d) this closed ball is compact (Remark 2.2 and Ballmann (1995), Thm. 2.4). Hence, given any sequence $(s_k)_k \subset (2^{-k})_k$ there exists a subsequence $(\delta_k)_k = (s_{n_k})_k$ such that $(z_{\delta_k})_k$ converges in N.

(b) Given $t \in \mathbb{T}$ and $x \in M$, the space $L_1((M, \mathcal{M}, p_t(x,.)), (\mathbb{R}, |.|))$ is known to be separable since $(M, \mathcal{M})$ is a Radon measurable space. Similarly, since the target (N,d) is separable, one verifies that the space $L_1((M, \mathcal{M}, p_t(x,.)), (N,d))$ is separable. Moreover, according to Lemma 3.1 (i), $d(z_s(t,x,f), z_s(t,x,g)) \le d_1(f,g)$ for all s and f,g under consideration. Hence, the subsequence $(\delta_k)_k$ in (a) can be chosen in such a way that $(z_{\delta_k}(t,x,f))_k$ converges in N for all $f \in L_1((M, \mathcal{M}, p_t(x,.)), (N,d))$. Due to condition (17), the latter contains the space $\mathrm{Lip}_{\rho_0}(M,N)$:

$$\int d(f(x), f(y)) p_t(x, dy) \le \mathrm{dil}_{\rho_0} f \cdot \int \rho_0(x,y) p_t(x, dy) < \infty.$$

(c) Let M_0 be a ρ-dense subset of M. Then the subsequence $(\delta_k)_k$ in (b) can be chosen in such a way that $(z_{\delta_k}(t,x,f))_k$ converges in N for all $t \in \mathbb{T}$, all $x \in M_0$ and all $f \in \mathrm{Lip}_{\rho_0}(M,N)$. Due to Lemma 3.1 and condition (16)

$$d(z_s(t,x,f), z_s(t,y,f)) \le p_t^\diamond d_f(x,y) \le \mathrm{dil}_{\rho_0} f \cdot p_t^\diamond \rho_0(x,y) \le C_t \cdot \mathrm{dil}_{\rho_0} f \cdot \rho(x,y)$$

for all s,t,x,y,f under consideration. Hence, $P_t^* f(x) = \lim_{k\to\infty} P_{\delta_k}^{t/\delta_k} f(x)$ exists for all $t \in \mathbb{T}, x \in M$, $f \in \mathrm{Lip}_{\rho_0}(M,N)$ and

$$d(P_t^* f(x), P_t^* f(y)) \le C_t \cdot \mathrm{dil}_{\rho_0} f \cdot \rho(x,y).$$

(d) Finally, Lemma 3.1(ii), (c) from above, and the definition of γ_r imply

$$\begin{aligned} &d(z_s(t,x,f), z_s(t+r,x,f)) \\ &\quad \le \int d(z_s(t,x,f), z_s(t,z,f)) p_r(x,dz) \le C_t \cdot \mathrm{dil}_{\rho_0} f \cdot \gamma_r(x) \end{aligned}$$

which yields the claim of the above Remark (ii). □

Example 4.5. Let $(q_t)_{t>0}$ be a convolution semigroup of probability measures on an Abelian group M and define a translation invariant Markov semigroup by $p_t(x,A) := q_t(x^{-1}A)$. Then for each symmetric $\rho : M \times M \to \mathbb{R}_+$

$$p_t^\diamond \rho(x,y) \le \int \rho(xz, yz)\, q_t(dz).$$

In particular, if ρ is translation invariant then

$$p_t^\diamond \rho(x,y) \le \rho(x,y).$$

For instance, this applies to all Lévy semigroups on $\mathbb{R}^n$ as introduced in Example 1.3. For various other examples, see Bendikov (1995), Bendikov, Saloff-Coste (2001) and Bloom, Heyer (1995).

Example 4.6. (i) Let $(p_t)_{t>0}$ be a Markov semigroup on $M = \mathbb{R}^n$ such that

$$p_t(x, B) = \int_{\mathbb{R}^n} 1_B(y) k_t(\|x - y\|) dy$$

for all $t > 0$ with some decreasing function $r \mapsto k_t(r)$ on $\mathbb{R}_+$ with

$$C_t := \int_{\mathbb{R}^{n-1}} k_t(\|z\|) dz < \infty. \tag{20}$$

(Note that here $z \in \mathbb{R}^{n-1}$ whereas before $y \in \mathbb{R}^n$.) Put $\rho(x, y) = \|x - y\|$. Then

$$p_t^\diamond 1(x, y) \le C_t \cdot \rho(x, y).$$

(ii) For instance, for $\alpha \le 2$ let $(p_t)_{t>0}$ be the *symmetric α-stable semigroup* on $\mathbb{R}^n$, i.e. the Markov semigroup associated with the Lévy operator $-(-\frac{1}{2}\Delta)^{\alpha/2}$. Then

$$C_t = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-t \cdot (|s|/\sqrt{2})^\alpha) ds = C'_\alpha \cdot t^{-1/\alpha}.$$

In particular, if $\alpha = 2$ then $(p_t)_{t>0}$ is the classical heat semigroup on $\mathbb{R}^n$, i.e. $k_t(r) = (2\pi t)^{-n/2} \exp(-r^2/(2t))$, and

$$C_t = \frac{1}{\sqrt{2\pi t}}.$$

(iii) More generally, let

$$k_t(\|z\|) = (2\pi)^{-n/2} \cdot \int_{\mathbb{R}^n} \exp(iz\xi) \cdot \exp(-t \cdot \Psi(\|\xi\|^2/2))\, d\xi$$

for $z \in \mathbb{R}^n$ with a Lévy function function Ψ on $\mathbb{R}_+$ (satisfying $\Psi(0) = 0$), corresponding to the Markov semigroup with generator $a = -\Psi(-\frac{1}{2}\Delta)$. Then

$$C_t = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-t \cdot \Psi(|s|^2/2)) ds. \tag{21}$$

Proof. (i) Put $u_0(x) := \frac{1}{2}\mathrm{sgn}(x_1)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and define $u_t := p_t u_0$ for $t > 0$. By symmetry of u_0 and p_t, there exists a function $\rho_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$u_t(x) = \frac{1}{2}\mathrm{sgn}(x_1)\rho_t(2|x_1|)$$

for all $x \in \mathbb{R}^n$.

Our first claim is

$$p_s^\triangledown \rho_t \le \rho_{s+t} \tag{22}$$

for all $s, t > 0$. In order to prove (22), fix $x, y \in \mathbb{R}^n$, $s, t > 0$ and $u \in \mathrm{Lip}_{\rho_t}(M, N)$ with $\mathrm{dil}_{\rho_t} u \le 1$. Without restriction, we may assume $x_1 \ge 0$ and $y = x^*$ where

$$z = (z_1, z_2, \ldots, z_n) \mapsto z^* := (-z_1, z_2, \ldots, z_n)$$

denotes the mirror map. Then

$$\begin{aligned}
p_s u(x) - p_s u(x^*) &= \int [u(z) - u(z^*)]\, k_s(\|z-x\|)dz \\
&= \frac{1}{2}\int [u(z) - u(z^*)] \cdot [k_s(\|z-x\|) - k_s(\|z+x\|)]\, dz \\
&\le \frac{1}{2}\int \rho_t(2|z_1|) \cdot \mathrm{sgn}(z_1) \cdot [k_s(\|z-x\|) - k_s(\|z+x\|)]\, dz \\
&= \int \rho(2|z_1|) \cdot \mathrm{sgn}(z_1) \cdot k_s(\|z-x\|)dz \\
&= 2u_{s+t}(x) = \rho_{s+t}(2x_1) = \rho_{s+t}(\|x-x^*\|).
\end{aligned}$$

The inequality in the above calculation holds true because $\mathrm{sgn}(z_1)\,[k_s(\|z-x\|) - k_s(\|z+x\|)] \ge 0$ since by assumption $x_1 \ge 0$ and since $r \mapsto k_s(r)$ is decreasing. This proves the claim (22).

By iteration, (22) implies $p_s^\diamond \rho_t \le \rho_{s+t}$ for all $s, t > 0$, in particular, $p_s^\diamond \rho_0 \le \rho_s$. Finally, note that

$$\begin{aligned}
\rho_s(r) &= 2u_s(+\frac{r}{2}, 0, \ldots, 0) = p_s(]-r/2, r/2[\times \mathbb{R}^{n-1}) \\
&= \int_{-r/2}^{r/2} \int_{\mathbb{R}^{n-1}} k_s((\xi, z))dzd\xi \le r \cdot \int_{\mathbb{R}^{n-1}} k_s((0,z))dz = r \cdot C_s.
\end{aligned}$$

Hence, $p_s^\diamond \rho_0 \le \rho_s \le C_s \cdot \rho$.

(ii), (iii) For $\varepsilon > 0$ and $z = (z', z_n) \in \mathbb{R}^n$ let $\phi_\varepsilon(z) = \varepsilon^{-(n-1)/2} \cdot \exp(-|z_n|^2\varepsilon/2) \cdot \exp(-\|z'\|^2/(2\varepsilon))$ and $k_t(z) := k_t(\|z\|)$. Then the respective Fourier transforms are $\hat{\phi}_\varepsilon(z) = \varepsilon^{-1/2} \cdot \exp(-|z_n|^2/(2\varepsilon)) \cdot \exp(-\|z'\|^2\varepsilon/2)$ and $\hat{k}_t(z) = (2\pi)^{-n/2} \cdot \exp(-t\Psi(\|z\|^2/2))$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} k_t(\|z'\|)dz' \\
&= \lim_{\varepsilon\to 0} (2\pi)^{-1/2} \int_{\mathbb{R}^n} \hat{\phi}_\varepsilon(z) k_t(z)\, dz \\
&= \lim_{\varepsilon\to 0} (2\pi)^{-1/2} \int_{\mathbb{R}^n} \phi_\varepsilon(z) \hat{k}_t(z)\, dz \\
&= \lim_{\varepsilon\to 0} (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} \phi_\varepsilon(z) \exp(-t\Psi(\|z\|^2/2))dz \\
&= (2\pi)^{-1} \int_{\mathbb{R}} \exp(-t\Psi(|z_n|^2/2))\, dz_n.
\end{aligned}$$

If $\Psi(r) = r^{\alpha/2}$, the last integral can be written as $t^{-1/\alpha} \cdot (2\pi)^{-1} \int_{\mathbb{R}} \exp(-|r/\sqrt{2}|^\alpha)\, dr$. $\square$

Example 4.7. Let $(p_t)_{t>0}$ be the heat semigroup on a complete Riemannian manifold M and let ρ be the Riemannian distance on M. Then for any number $\kappa \in \mathbb{R}$ the following are equivalent (von Renesse, Sturm (2004)):

(i) $\mathrm{Ric}_M(\xi, \xi) \ge -\kappa \cdot |\xi|^2$ for all $\xi \in TM$ (briefly: $\mathrm{Ric}_M \ge -\kappa$);

(ii) $p_t^\diamond \rho(x, y) \le e^{\kappa t} \cdot \rho(x, y)$ for all $x, y \in M$.

Moreover, in this case, there exist $C = C(\kappa, n)$ such that $\forall x, y, t :$

- $p_t^\diamond 1(x, y) \le C \cdot t^{-1/2} \cdot e^{\kappa t} \cdot \rho(x, y)$

- $\int \rho^4(x,y) p_t(x,dy) \leq C \cdot t^2$.

For examples of Alexandrov spaces M (= complete metric spaces of lower bounded "sectional" curvature) where the same estimates hold true, see VON RENESSE (2002).

Example 4.8. Let M and $(p_t)_{t>0}$ as in the previous Example 4.7 and fix an open subset $D \subset M$. Let $(p_{D,t})_{t>0}$ be the stopped semigroup as introduced in Definition 1.6. Then for this semigroup,

$$p^{\diamond}_{D,t} 1 \leq C \cdot \rho \qquad \text{on } B \times B$$

with $C = C(t, B)$ for each open set B which is relatively compact in D.

In Chapter 6 we will see that this implies local Lipschitz continuity on D for each map $f : M \to N$ which is harmonic on D. For a similar condition which implies Hölder continuity at the boundary, see Remark 6.13 below.

Example 4.9. Let $(p_t)_{t>0}$ be a strongly continuous, symmetric semigroup on a σ-finite measure space $(M, \mathcal{M}, m)$ and assume a "curvature-dimension condition" in the sense of Bakry-Emery (see e.g. LEDOUX (2000)) holds true with curvature bound $-\kappa$ and dimension bound n. Moreover, let ρ be a symmetric nonnegative function on $M \times M$ with the "Rademacher property"

$$\mathrm{dil}_\rho u \leq 1 \quad \Longleftrightarrow \quad u \in \mathcal{D}(\Gamma), \Gamma(u) \leq 1 \ m - \text{a.e.}$$

where Γ denotes the square field operator associated with $(p_t)_{t>0}$. Then

$$p^{\diamond}_t \rho(x,y) \leq e^{\kappa t} \cdot \rho(x,y).$$

For instance, this applies to the Ornstein-Uhlenbeck semigroup on the Wiener space $M = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Here $m =$ Wiener measure, $\rho =$ Cameron-Martin distance, $-\kappa = 1$ and $n = \infty$.

Example 4.10. Let (M, ρ, m) be a metric measure space and define for $r > 0$ the kernel $q_r(x, dy)$ of uniform distribution in the ball of radius r by

$$q_r(x, A) = \frac{m(A \cap B_r(x))}{m(B_r(x))}.$$

Assume that there exists a number $\kappa \in \mathbb{R}$ such that

$$\rho^W(q_r(x_1,.), q_r(x_2,.)) \leq \left[1 + \kappa r^2 + o(r^2)\right] \cdot \rho(x_1, x_2) \tag{23}$$

for all $x_1, x_2 \in M$ and $r \to 0$. Then each null sequence $(r_n)_{n \in \mathbb{N}}$ for which

$$p_t u(x) = \lim_{n \to \infty} (q_{r_n})^{\lfloor t/r_n^2 \rfloor} u(x)$$

exists (for all $x \in M, t > 0$ and bounded $u \in \mathrm{Lip}(M)$), it defines a Markov semigroup $(p_t)_t$ satisfying

$$p^{\diamond}_t \rho(x,y) \leq e^{\kappa t} \cdot \rho(x,y).$$

(If M is separable, one always will find such a sequence for which the convergence is guaranteed, cf. proof of Theorem 4.3.)

For a Riemannian manifold M equipped with its Riemannian distance ρ and its Riemannian volume measure m, condition (23) is *equivalent* to

$$\mathrm{Ric}_M \geq -c \cdot \kappa$$

with $c = 1/\sqrt{2(n+2)}$ (VON RENESSE, STURM (2004)). The Markov semigroup constructed as above as scaling limit of the q_r is just the heat semigroup (rescaled by the factor c):

$$p_t = \exp(ct\Delta).$$

Example 4.11. For $k \in \mathbb{N}$, let M be the metric completion of the k-fold cover of $\mathbb{R}^2 \setminus \{0\}$ equipped with the metic $\rho(x,y) = \|x-y\|^{1/k}$. Moreover, let $(p_t)_t$ be the semigroup for the generator $a = \|x\|^{2k-2} \cdot \Delta$. Then

$$p_t^{\diamond}\rho(x,y) \le \rho(x,y).$$

In terms of the Euclidean metric this means that our generalized harmonic maps will be Hölder continuous with exponent $1/k$. This is best possible since even harmonic functions (like $x = (r,\varphi) \mapsto r^{1/k} \cdot \cos(\varphi/k)$) will have no better continuity properties.

5 L_θ-Contraction Properties

In the previous Chapters, we have presented the pointwise approach to nonlinear Markov operators and nonlinear heat semigroups. In this Chapter, we present the uniform and the L_θ-approach. As before (N,d,b) will be a complete metric space with barycenter contraction and $(M,\mathcal{M})$ will be a measurable space with a Markov semigroup $(p_t)_{t>0}$ on it. Let us firstly have a brief look on the uniform approach. Let $\mathcal{L}_\infty(M,N,p)$ denote the set of measurable $f : M \to N$ with separable range $f(M)$ and with bounded $\eta_t f$ (for each $t>0$). And let $\mathcal{L}^*_\infty(M,N,p)$ denote the set of $f \in \mathcal{L}_\infty(M,N,p)$ for which the *uniform limit*

$$P_t^* f := \lim_{n\to\infty} (P_{\delta_n})^{t/\delta_n} f$$

exists for all $t \in \mathbb{T}$. Then $(P_t^*)_{t\in\mathbb{T}}$ will be a contraction semigroup on $\mathcal{L}^*_\infty(M,N,p)$ (equipped with the uniform distance). This and further results will be deduced in the following more general framework.

In addition to the previous, we now fix a measure m on $(M,\mathcal{M})$ and a number $\theta \in [1,\infty]$ and we assume that there exists constants $C, \alpha \in \mathbb{R}$ such that

$$\|p_t u\|_\theta \le C \cdot e^{\alpha t} \cdot \|u\|_\theta \tag{24}$$

for all bounded measurable $u : M \to \mathbb{R}$ and all $t > 0$. In other words, we assume that $(p_t)_t$ extends to an exponentially bounded semigroup on $L_\theta(M) := L_\theta(M,\mathcal{M},m)$, the Lebesgue space of m-equivalence classes of measurable functions $u : M \to \mathbb{R}$.

Example 5.1. (i) Let $(p_t)_{t>0}$ be any Markov semigroup on a measurable space $(M,\mathcal{M})$. Choose $m := \sum_{x\in M} \delta_x$ to be the counting measure. Then for all measurable $u : M \to \mathbb{R}$ and all $t > 0$

$$\|p_t u\|_\infty \le \|u\|_\infty.$$

Hence, without any restriction the uniform norm is always included in the follwing discussions as an L_∞-norm (trivially satisfying (24)).

(ii) Each semigroup $(p_t)_{t>0}$ derived from a symmetric Dirichlet form on $L_2(M,\mathcal{M},m)$ satisfies $\|p_t u\|_\theta \le \|u\|_\theta$ for each $\theta \in [1,\infty]$. For $\theta = 2$ we even obtain

$$\|p_t u\|_2 \le e^{\alpha t} \cdot \|u\|_2$$

with $\alpha = \sup \operatorname{spec}(a) = -\inf_{u \in L_2(M)} \mathcal{E}(u)/\|u\|_2^2 \le 0$ being the top of the L_2-spectrum of the generator $a = \lim_{t\to 0} \frac{1}{t}(p_t - 1)$ of the Dirichlet form.

For measurable $f, g : M \to N$ put $d_\theta(f,g) := \|d(f,g)\|_\theta$ where $d(f,g)$ denotes the function $x \mapsto d(f(x), g(x))$ on M. In particular, for $\theta < \infty$

$$d_\theta(f,g) = \left(\int_M d^\theta(f(x), g(x))\, m(dx) \right)^{1/\theta}.$$

Let $L_\theta(M,N,p)$ denote the set of equivalence classes of measurable $f : M \to N$ with separable ranges and with $\eta_t f \in L_\theta(M)$ for all $t > 0$. One easily verifies that if m is a finite measure then $(L_\theta(M,N,p), d_\theta)$ is a complete metric space and each constant map lies in $L_\theta(M,N,p)$. Moreover, for each measurable $g : M \to N$ with separable range

$$f \in L_\theta(M,N,p),\ d_\theta(f,g) < \infty \Longrightarrow g \in L_\theta(M,N,p).$$

If $\tilde{f}$ is a fixed version of $f \in L_\theta(M,N,p)$ and $t > 0$ then $\eta_t \tilde{f}(x) < \infty$ for m-a.e. $x \in M$. Hence, $P_t \tilde{f}(x) := b(p_t(x, \tilde{f}^{-1}(.)))$ is well-defined for m-a.e. $x \in M$ and according to Lemma 3.1 (i) and assumption (24)

$$d_\theta(P_t \tilde{f}, P_t \tilde{g}) \le C \cdot e^{\alpha t} d_\theta(f,g)$$

for any version $\tilde{g}$ of another $g \in L_\theta(M,N,p)$. Let $\overline{P}_t f$ denote the m-equivalence class of $P_t \tilde{f}$ (which by the preceding only depends on the class f, not on the particular choice of the version $\tilde{f}$). Then

$$f \in L_\theta(M,N,p) \Longrightarrow \overline{P}_t f \in L_\theta(M,N,p) \quad \text{and} \quad d_\theta(f, \overline{P}_t f) \le \|\eta_t f\|_\theta.$$

Let $L^*_\theta(M,N,p)$ denote the set of $f \in L_\theta(M,N,p)$ for which the d_θ-limit

$$\overline{P}^*_t f := \lim_{n \to \infty} (\overline{P}_{\delta_n})^{t/\delta_n} f$$

exists for all $t \in \mathbb{T}$.

Example 5.2. Let $N = \mathbb{R}$ (with the usual d and b) and let $(p_t)_{t>0}$ be the heat semigroup on $M = \mathbb{R}^1$ (with m being the Lebesgue measure). Then $L_\theta(M,N,p) = L^*_\theta(M,N,p) \supset L_\theta(M)$ with strict inclusion. Indeed, consider the function $f(x) = (1 + |x|)^\alpha$. Then for $\theta < \infty$

$$f \in L_\theta(M) \Longleftrightarrow \alpha < -1/\theta \quad \text{and} \quad f \in L_\theta(M,N,p) \Longleftrightarrow \alpha < 1 - 1/\theta$$

(since $\eta_t f(x) \approx C \cdot \alpha \cdot \sqrt{t} \cdot |x|^{\alpha - 1}$ for large x). Similarly,

$$f \in L_\infty(M) \Longleftrightarrow \alpha \le 0 \quad \text{and} \quad f \in L_\infty(M,N,p) \Longleftrightarrow \alpha \le 1.$$

With exactly the same arguments as for Lemma 3.2 we deduce

Lemma 5.3. *For all $f, g \in L^*_\theta(M,N,p)$ and all $s, t \in \mathbb{T}$:*

(i) $\overline{P}^*_t f \in L^*_\theta(M,N,p)$ and $\overline{P}^*_s(\overline{P}^*_t f) = \overline{P}^*_{s+t} f$;

(ii) $d_\theta(\overline{P}^*_t f, \overline{P}^*_t g) \le C \cdot e^{\alpha t} \cdot d_\theta(f,g)$;

(iii) $d_\theta(\overline{P}^*_t f, \overline{P}^*_{t+s} f) \le C \cdot e^{\alpha t} \cdot \|\eta_s f\|_\theta$.

Remark 5.4. (i) The set $L^*_\theta(M,N,p)$ is closed w.r.t. d_θ.
(ii) For all $f \in L^*_\theta(M,N,p)$ with $\lim_{t\to 0} \|\eta_t f\|_\theta = 0$ the map $t \mapsto \overline{P}^*_t f$ is continuous in $t \in \mathbb{T}$ (according to 5.3(iii)) and thus

$$\overline{P}^*_t f = \lim_{\mathbb{T}\ni s\to t} \overline{P}^*_s f = \lim_{n\to\infty} (\overline{P}_{\delta_n})^{\lfloor t/\delta_n \rfloor} f \ \in L^*_\theta(M,N,p)$$

is well-defined for all $t > 0$.
(iii) For each $\theta < \infty$

$$L^*_\theta(M,N,p) \supset L_\theta(M,N,p) \cap \mathcal{L}^*(M,N,p).$$

Indeed, $f \in L_\theta(M,N,p) \cap \mathcal{L}^*(M,N,p)$ implies $u_n := d(P^{t/\delta_n}_{\delta_n} f, P^*_t f) \longrightarrow 0$ pointwise on M for $n \to \infty$ and $u_n \le 2\eta_t f \in L_\theta(M)$ for all $\delta_n \le t$. Hence, by Lebesgue's dominated convergence theorem $u_n \to 0$ in $L_\theta(M)$ and thus $\overline{P}^{t/\delta_n}_{\delta_n} f \to \overline{P}^*_t f$ in $L_\theta(M,N,p)$.
(iv) Assume that $p_t(x,.) \ll m$ for all $x \in M$ and all $t \in \mathbb{T}$ ("absolute continuity of p_t"). Then $f(x) = g(x)$ for m-a.e. $x \in M$ implies $P^*_t f(x) = P^*_t g(x)$ for all $x \in M$ and $t \in \mathbb{T}$. In particular, for each $f \in L^*_\theta(M,N,p)$ and each $t \in \mathbb{T}$ the map $\overline{P}^*_t f$ is pointwise well-defined on M. In this case, there is no need to distinguish between P_t and $\overline{P}_t$.

Theorem 5.5. *Assume that for some $t \in \mathbb{T}$, $\theta' \in [1,\infty]$ and $C \in \mathbb{R}_+$*

$$\|p_t u\|_{\theta'} \le C \cdot \|u\|_\theta \qquad (\forall u \in L_\theta(M)).$$

Then

$$d_{\theta'}(\overline{P}^*_t f, \overline{P}^*_t g) \le C \cdot d_\theta(f,g) \qquad (\forall f,g \in L^*_\theta(M,N,p)).$$

Proof. Put $u : x \mapsto d(f(x),g(x))$. Then by Lemma 3.2 (i)

$$\begin{aligned} d_{\theta'}(\overline{P}^*_t f, \overline{P}^*_t g) &\le \left[\int_M \left(\int_M d(f(y), g(y)\, p_t(x,dy) \right)^{\theta'} m(dx) \right]^{1/\theta'} \\ &= \left[\int_M p_t u(x)^{\theta'} m(dx) \right]^{1/\theta'} \le \|p_t\|_{\theta,\theta'} \cdot \|u\|_\theta \\ &\le C \cdot d_\theta(f,g) \end{aligned}$$

□

As an immediate corollary we deduce that if the linear semigroup $(p_t)_{t>0}$ acting on $L_\theta(M), 1 \le \theta \le \infty$, is hyper-, ultra- or supercontractive then so is the nonlinear semigroup $(\overline{P}^*_t)_{t\in\mathbb{T}}$ acting on $L_\theta(M,N,p), 1 \le \theta \le \infty$. We quote the following main examples.

Corollary 5.6. *Given $t \in \mathbb{T}$, assume that the Markov kernel p_t has a bounded density $k_t(x,y) := \frac{p_t(x,dy)}{m(dy)} \le C_t$. Then for all $1 \le \theta \le \theta' \le \infty$*

$$d_{\theta'}(\overline{P}^*_t f, \overline{P}^*_t g) \le C_t^{1/\theta - 1/\theta'} \cdot d_\theta(f,g). \qquad (\forall f,g \in L^*_\theta(M,N,p))$$

Corollary 5.7. *Let the Markov semigroup $(p_t)_t$ be associated with a symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L_2(M)$.*

(i) Assume either that a "Nash inequality" holds true for $\mu > 0$ (with constants C_0, C_1):

$$\|u\|_2^{2+4/\mu} \le \left[C_0 \cdot \mathcal{E}(u) + C_1 \cdot \|u\|^2 \right] \cdot \|u\|_1^{4/\mu} \qquad (\forall u \in \mathcal{D}(\mathcal{E}))$$

or that a "Sobolev inequality" holds true for $\mu > 2$ (with constants C_0, C_1):

$$\|u\|^2_{2\mu/(\mu-2)} \leq C_0 \cdot \mathcal{E}(u) + C_1 \cdot \|u\|^2. \qquad (\forall u \in \mathcal{D}(\mathcal{E}))$$

Then for some constant C, all $t \in \mathbb{T}, t \leq 1$ (or even all $t \in \mathbb{T}$ if $C_1 = 0$) and all $1 \leq \theta \leq \theta' \leq \infty$

$$d_{\theta'}(\overline{P}^*_t f, \overline{P}^*_t g) \leq C \cdot t^{-\frac{\mu}{2}(\frac{1}{\theta}-\frac{1}{\theta'})} \cdot d_\theta(f,g). \qquad (\forall f, g \in L^*_\theta(M,N,p))$$

(ii) Assume that m is a probability measure and that a "logarithmic Sobolev inequality" with constant $\nu > 0$ holds true for all $u \in \mathcal{D}(\mathcal{E})$ with $\|u\|_2 = 1$:

$$\int_M u^2 \log u^2 \, dm \leq \frac{2}{\nu} \cdot \mathcal{E}(u).$$

Then for all $t \in \mathbb{T}$ and all $1 < \theta < \theta' < \infty$ with $\frac{\theta'-1}{\theta-1} < e^{2\nu t}$:

$$d_{\theta'}(\overline{P}^*_t f, \overline{P}^*_t g) \leq d_\theta(f,g). \qquad (\forall f, g \in L^*_\theta(M,N,p))$$

Cf. Davies (1989), Ledoux (2000).

Theorem 5.8. *Assume that $\alpha < 0$ in (24). Then for each $f \in L^*_\theta(M,N,p)$ there exists a unique $h \in L^*_\theta(M,N,p)$ with $d_\theta(h,f) < \infty$ and*

$$\overline{P}^*_t h = h \tag{25}$$

for all $t \in \mathbb{T}$. Indeed, $h = \lim_{\mathbb{T}\ni t\to\infty} \overline{P}^*_t f$ and for $t \to \infty$

$$d_\theta(h, \overline{P}^*_t f) \leq C \cdot e^{\alpha t} \cdot d_\theta(h,f) \longrightarrow 0.$$

Proof. Uniqueness is obvious from

$$d_\theta(h,h') = d_\theta(\overline{P}^*_t h, \overline{P}^*_t h') \leq C \cdot e^{\alpha t} \cdot d_\theta(h,h') \to 0$$

(as $t \to \infty$). For the existence, note that for all $\delta \in \mathbb{T}$ and $n \in \mathbb{N}$

$$d_\theta(\overline{P}^*_{n\delta} f, \overline{P}^*_{(n+1)\delta} f) \leq C \cdot e^{\alpha n \delta} \cdot \|\eta_\delta f\|_\theta.$$

Hence, $(\overline{P}^*_{n\delta} f)_{n\in\mathbb{N}}$ is a Cauchy sequence and, by completeness, there exists $h \in L_\theta(M,N,f)$ such that $h_\delta = \lim_{n\to\infty} \overline{P}^*_{n\delta} f$. Replacing δ by $\delta/2$ shows that $h_\delta = h_{\delta/2} =: h$. Thus $h = \lim_{\mathbb{T}\ni t\to\infty} \overline{P}^*_t f$ and $\overline{P}^*_t h = h$ for all $t \in \mathbb{T}$. □

The above result may be used to deduce existence and uniqueness of the solution to the Dirichlet problem. Namely, given a Markov semigroup $(p_t)_{t>0}$ on a complete separable metric space M and a bounded open subset $D \subset M$, let $(p_{D,t})_{t>0}$ be the stopped semigroup as introduced in Chapter 1 and replace the measure $m(dx)$ by $(1_D(x) + \infty \cdot 1_{M\setminus D}(x))m(dx)$. Then in most examples (due to the boundedness of D)

$$\|p_{D,t} u\|_\theta \leq C \cdot e^{\alpha_D t} \cdot \|u\|_\theta \tag{26}$$

with some $\alpha_D < 0$. Hence, for each $f \in L^*_\theta(M,N,p_D)$ there exists a unique $h \in L^*_\theta(M,N,p_D)$ with

- $d_\theta(h,f) < \infty$,
- $h = f$ m-a.e. on $M \setminus D$,
- $\overline{P}^*_{D,t} h = h$ for all $t \in \mathbb{T}$.

This map h will be a solution to the nonlinear Dirichlet problem (for the given domain D and the data f) as defined in the next Chapter.

6 Invariant and Harmonic Maps

Definition 6.1. A map $f : M \to N$ is called *invariant* iff $f \in \mathcal{L}^*(M, N, p)$ and for all $t \in \mathbb{T}$ and $x \in M$

$$P_t^* f(x) = f(x).$$

It is called *harmonic* in a point $x \in M$ iff $f \in \mathcal{L}^*(M, N, p)$ and $A^* f(x) = 0$ where

$$A^* f(x) := \limsup_{\mathbb{T} \ni t \to 0} \frac{1}{t} d(f(x), P_t^* f(x)).$$

Obviously, a map f is invariant if and only if $f \in \mathcal{L}(M, N, p)$ and for all $t \in \mathbb{T}$ and $x \in M$

$$\limsup_{n \to \infty} d(f(x), P_{\delta_n}^{t/\delta_n} f(x)) = 0$$

and, of course, each invariant map is harmonic on M.

We say that a function $u : M \to \mathbb{R}$ is *subinvariant* iff $u \in \mathcal{L}(M, \mathbb{R}, p)$ and $u(x) \leq p_t u(x)$ for all all $t \in \mathbb{T}$ and $x \in M$ and we say that it is *subharmonic* iff $u \in \mathcal{L}(M, \mathbb{R}, p)$ and $\underline{a} u(x) \geq 0$ where

$$\underline{a} u(x) := \liminf_{\mathbb{T} \ni t \to 0} \frac{1}{t} (p_t u(x) - u(x)).$$

Remark 6.2. Let $(p_t)_t$ be the classical heat semigroup on a Riemannian manifold M. One easily verifies that $\underline{a} u(x) = \frac{1}{2} \Delta u(x)$ for each $x \in M$ and each bounded real valued function u which is smooth in a neighborhood of x.

More generally, *for any open set $D \subset M$ and for any bounded, upper semicontinuous function $u : M \to \mathbb{R}$ the following properties are equivalent:*

(i) $\underline{a} u(x) \geq 0$ *for all $x \in D$;*

(ii) $\Delta u \geq 0$ *on D in distributional sense;*

(iii) *u is subharmonic on D in the classical sense;*

(iv) $p_{B,t} u \geq u$ *on M for each open set B which is relatively compact in D and each $t > 0$.*

Moreover, in (i) it suffices that $\underline{a} u \geq 0$ m-a.e. on D, and in (iv) it suffices to consider balls B which are relatively compact in D.

Proof. (i) $\Rightarrow$ (ii): Using Fatou's lemma and the symmetry of p_t, we conclude for each $\psi \in \mathcal{C}_c^\infty(D)$

$$\begin{aligned}
0 \quad &\leq \quad \int_M \underline{a} u(x) \cdot \psi(x)\, dx \;=\; \int_M \liminf_{t \to 0} \frac{1}{t} [p_t u(x) - u(x)] \cdot \psi(x)\, dx \\
&\leq \quad \liminf_{t \to 0} \int_M \frac{1}{t} [p_t u(x) - u(x)] \cdot \psi(x)\, dx \;=\; \liminf_{t \to 0} \int_M \frac{1}{t} [p_t \psi(x) - \psi(x)] \cdot u(x)\, dx \\
&\leq \quad \int_M \lim_{t \to 0} \frac{1}{t} [p_t \psi(x) - \psi(x)] \cdot u(x)\, dx \;=\; \int_M \frac{1}{2} \Delta \psi(x) \cdot u(x)\, dx.
\end{aligned}$$

This proves that $\frac{1}{2} \Delta u \geq 0$ on D in distributional sense.

(ii) $\Rightarrow$ (iii): See e.g. Hörmander (1990).

(iii) $\Rightarrow$ (iv): Classical potential theory (or Ito's formula).

(iv) $\Rightarrow$ (i): Obviously, (iv) implies that $\underline{a}_B u(x) := \liminf_{t\to 0} \frac{1}{t}[p_{B,t}u(x) - u(x)] \geq 0$ for each $x \in B$. According to Proposition 6.9 and Example 6.10 below this is equivalent to $\underline{a}u(x) \geq 0$ for each $x \in B$. $\square$

A simple consequence of Jensen's inequality is

Proposition 6.3. *Let (N,d) be either a complete metric space of globally nonpositive curvature or a Banach space and let $\varphi : N \to \mathbb{R}$ be convex and Lipschitz continuous.*
(i) If $f \in \mathcal{L}^*(M,N,p)$ is invariant then $\varphi \circ f : M \to \mathbb{R}$ is subinvariant.
(ii) If f is harmonic on some set D then $\varphi \circ f : M \to \mathbb{R}$ is subharmonic on this set D.

Proof. Firstly, observe that $\eta_t(\varphi \circ f)(x) \leq \mathrm{dil}(\varphi) \cdot \eta_t f(x)$. Thus $f \in \mathcal{L}(M,N,p)$ implies $\varphi \circ f \in \mathcal{L}(M,\mathbb{R},p)$.

Secondly, by Jensen's inequality (see Eells, Fuglede (2001) in the case of NPC spaces and e.g. Ledoux, Talagrand (1991) in the case of Banach spaces)

$$p_t(\varphi \circ f)(x) \geq \varphi(P_t f)(x)$$

for all Lipschitz continuous convex φ. By iteration $p_t(\varphi\circ f)(x) \geq \varphi(P_{\delta_n}^{t/\delta_n} f)(x)$ and $p_t(\varphi\circ f)(x) \geq \varphi(P_t^* f)(x)$ provided $f \in \mathcal{L}^*(M,N,p)$. Hence, invariance of f implies $p_t(\varphi \circ f)(x) \geq \varphi(f)(x)$. And harmonicity of f implies

$$\begin{aligned}
\underline{a}(\varphi \circ f)(x) &= \liminf_{t\to 0} \frac{1}{t}\left[p_t(\varphi \circ f)(x) - \varphi(f)(x)\right] \\
&\geq \liminf_{t\to 0} \frac{1}{t}\left[\varphi(P_t^* f)(x) - \varphi(f)(x)\right] \\
&\geq -\mathrm{dil}(\varphi) \cdot \limsup_{t\to 0} \frac{1}{t} d\left(P_t^* f(x), f(x)\right) \\
&= -\mathrm{dil}(\varphi) \cdot A^* f(x) = 0.
\end{aligned}$$

$\square$

A general uniqueness result for the Dirichlet problem may be deduced from the following Proposition.

Proposition 6.4. *(i) If $f,g : M \to N$ are invariant then $u : x \mapsto d(f(x),g(x))$ is subinvariant.*
(ii) If maps f,g are harmonic in $x \in M$ then the function $u := d(f,g)$ is subharmonic in x.

Proof. By triangle inequality, $\eta_t u(x) \leq \eta_t f(x) + \eta_t g(x)$. Hence, $u \in \mathcal{L}(M,\mathbb{R},p)$. Moreover, by Lemmas 3.1(i) and 3.2(i)

$$\begin{aligned}
p_t u(x) - u(x) &= \int d(f(y),g(y)) p_t(x,dy) - d(f(x),g(x)) \\
&\geq d(P_t^* f(x), P_t^* g(x)) - d(f(x),g(x)) \geq -d(P_t^* f(x), f(x)) - d(P_t^* g(x), g(x)).
\end{aligned}$$

This proves the claims. $\square$

Remark 6.5. (i) We could call a map f *pseudo harmonic* in a point x iff $f \in \mathcal{L}(M,N,p)$ and $Af(x) = 0$ where

$$Af(x) := \limsup_{\mathbb{T}\ni t\to 0} \frac{1}{t} d(f(x), P_t f(x)).$$

Then most of the previous results for harmonic maps also hold true for pseudo harmonic maps: If φ is Lipschitz continuous convex function and if maps f and g are pseudo harmonic in x then the functions $d(f,g)$ and $\varphi(f)$ are subharmonic in x (Propositions 6.4 and 6.3). And finally in the framework of Proposition 6.9, $A_D f(x) = Af(x)$.

Unfortunately, however, in general there is no relation between pseudo harmonic and invariant maps. For a different situation in the uniform and L_θ-case, see (ii) below.

(ii) Let us assume the $\mathcal{L}_\infty$-framework or, more generally, the L_θ-framework of the previous Chapter. We will say that a map f is L_θ-invariant iff $f \in L^*_\theta(M,N,p)$ and $\overline{P}^*_t f = f$ for all $t \in \mathbb{T}$. It will be called L_θ-harmonic iff $f \in L^*_\theta(M,N,p)$ and $A^*_\theta f = 0$ where $A^*_\theta f := \limsup_{\mathbb{T}\ni t\to 0} \frac{1}{t} d_\theta(f, \overline{P}^*_t f)$. However, it turns out that L_θ-invariance and L_θ-harmonicity are the same. Indeed, since $(\overline{P}^*_t)_t$ is a semigroup on $L^*_\theta(M,N,p)$, one easily verifies the implications

$$(a) \quad \Longrightarrow \quad (b) \quad \Longleftrightarrow \quad (c)$$

for the statements below

(a) $f \in L_\theta(M,N,p)$ and $\lim\limits_{t\to 0} \frac{1}{t} d_\theta(f, \overline{P}_t f) = 0$;

(b) $f \in L^*_\theta(M,N,p)$ and $\lim\limits_{t\to 0} \frac{1}{t} d_\theta(f, \overline{P}^*_t f) = 0$;

(c) $f \in L^*_\theta(M,M,p)$ and $\overline{P}^*_t f = f$.

(iii) If $(p_t)_t$ is absolutely continuous w.r.t. the measure m (see Remark 5.4 (iv)) then each L_θ-invariant map is already invariant (more precisely, it admits an invariant version, see Sturm (2001), Prop. 6.2).

In order to formulate and solve the Dirichlet problem, let us assume for the rest of this Chapter that $(p_t)_t$ is a right Markov semigroup on a complete separable metric space M. Given an open subset D of M, let $(p_{D,t})_t$ always denote the stopped semigroup as introduced in Chapter 1 and let $(P^*_{D,t})_t$ be the nonlinear semigroup (acting on maps) derived from it.

Definition 6.6. Given an open set $D \subset M$ and a map $f : M \to N$, we say that g is a solution to the nonlinear *Dirichlet problem* iff $g = P^*_{D,t} g$ (for all $t \in \mathbb{T}$) and $g = f$ on $M \setminus D$.

Proposition 6.4 and Theorem 5.8 may be used to deduce existence and uniqueness for the solution to the Dirichlet problem. For sake of simplicity, we restrict ourselves to the pointwise version (with uniform convergence). Similar results hold true in $L_\theta(M,N,p)$. For simplicity, we also assume in the sequel that D is regular.

Corollary 6.7. *Assume that*

$$\mathbb{P}(\tau(D,x) < \infty) = 1$$

(for all $x \in M$) or, equivalently, that the following Maximum Principle holds true:
if $u : M \to \mathbb{R}_+$ is bounded, subinvariant for $(p_{D,t})_t$ and vanishes on $M \setminus D$ then $u = 0$.
Then bounded solutions to the nonlinear Dirichlet problem for harmonic maps on D are unique.

Corollary 6.8. *Assume that for some $t_0 > 0$*

$$\sup_{x\in D} \mathbb{P}(\tau(D,x) > t_0) < 1. \tag{27}$$

Then for each bounded $f \in \mathcal{L}^*(M,N,p_D)$ there exists a unique $g \in \mathcal{L}^*(M,N,p_D)$ with $d_\infty(g,f) < \infty$ and $g = f$ on $M \setminus D$ and $P^*_{D,t} g = g$ for all $t \in \mathbb{T}$. Namely, $g = \lim\limits_{\mathbb{T}\ni t\to\infty} P^*_{D,t} f$.

Proof. We may regard $(p_{D,t})_t$ as a contraction semigroup on the set of bounded measurable functions $u : M \to \mathbb{R}$ which vanish on $M \setminus D$, equipped with the uniform norm. By assumption, D is open and $(X_t^x)_t$ is right continuous. Therefore, $\mathbb{P}(X^x_{\tau(D,x)} \in D) = 0$ for all x and thus the norm of the operator $p_{D,t}$ can be expressed as follows

$$\|p_{D,t}\| = \sup_{x\in D} p_{D,t}(x, D) = \sup_{x\in D} \mathbb{P}(\tau(D, x) > t) \leq 1.$$

Moreover, for all $t > 0$

$$\|p_{D,t}\| \leq \|p_{D,t_0}\|^{\lfloor t/t_0\rfloor} \leq C \cdot e^{\alpha t}$$

with some $\alpha < 0$ provided $\|p_{D,t_0}\| < 1$ for some t_0. The claim follows now as in the proof of Theorem 5.8. □

In typical examples, condition (27) is fulfilled for each bounded open subset $D \subset M$.

Let us note that our solution of the Dirichlet problem will be harmonic in D, provided the underlying linear semigroup is *local* (in a suitable sense).

Proposition 6.9. *Let $(p_t)_t$ be a right Markov semigroup and assume that for given $D \in \mathcal{M}$ and $x \in D$ and the following locality condition is satisfied:*

$$\lim_{\mathbb{T}\ni t\to 0} \frac{1}{t}\mathbb{P}(\tau(D, x) < t) = 0. \tag{28}$$

Then a bounded map $f \in \mathcal{L}^*(M, N, p) \cap \mathcal{L}^*(M, N, p_D)$ is harmonic in x w.r.t. the semigroup $(p_t)_t$ if and only if it is harmonic in x w.r.t. the stopped semigroup $(p_{D,t})_t$. In particular, $P^*_{D,t}f(x) = f(x)$ for all $t \in \mathbb{T}$ implies $A^*f(x) = 0$.

Proof. Since the measure $\mathbb{P}\left((X^x_t, X^x_{t\wedge\tau(D,x)}) \in .\right)$ is a coupling of $p_t(x, .)$ and $p_{D,t}(x, .)$, the contraction property of barycenters implies for all t, x, f, g

$$d(P_tf(x), P_{D,t}g(x)) \leq d^W\left(p_t(x, f^{-1}(.)), p_{D,t}(x, g^{-1}(.))\right) \leq \mathbb{E}d(f(X^x_t), g(X^x_{t\wedge\tau(D,x)}))$$

and by iteration

$$d(P^n_{t/n}f(x), P^n_{D,t/n}g(x)) \leq \mathbb{E}d(f(X^x_t), g(X^x_{t\wedge\tau(D,x)}))$$

for all $n \in \mathbb{N}$. Hence,

$$d(P^*_tf(x), P^*_{D,t}g(x)) \leq \mathbb{E}d(f(X^x_t), g(X^x_{t\wedge\tau(D,x)}))$$

and

$$\begin{aligned}&\left|\frac{1}{t}d\left(f(x), P^*_tf(x)\right) - \frac{1}{t}d\left(f(x), P^*_{D,t}f(x)\right)\right| \\ &\leq\ \frac{1}{t}d\left(P^*_tf(x), P^*_{D,t}f(x)\right)\ \leq \frac{1}{t}\mathbb{E}d\left(f(X^x_t), f(X^x_{t\wedge\tau(D,x)})\right)\ \leq\ \mathrm{osc}(f)\cdot\frac{1}{t}\mathbb{P}(\tau(D, x) < t)\end{aligned}$$

where $\mathrm{osc}(f) := \sup_{y,z\in M} d(f(y), f(z))$. This proves the claim. □

Example 6.10. Let $(p_t)_t$ be the heat semigroup on a Riemannian manifold M (with arbitrary "boundary conditions at infinity") and D be an open subset of M. Then (28) is satisfied for all $x \in D$. Indeed, choose $r > 0$ such that $B_r(x) \subset D$. Then

$$\frac{1}{t}\mathbb{P}(\tau(D, x) < t) \leq \frac{1}{t}\mathbb{P}\left(\sup_{0\leq s\leq t} d(X^x_s, x) > r\right) \approx \frac{1}{t}\mathbb{P}\left(\sup_{0\leq s\leq t} |W_s| > r\right) \leq \frac{2n}{t}\mathbb{P}\left(\sup_{0\leq s\leq t} W^1_s > \frac{r}{\sqrt{n}}\right)$$

for $t \ll 1$ where $(W_s)_s$ denotes a Brownian motion on $\mathbb{R}^n$, starting in 0, with coordinate processes $(W_s^1)_s, \ldots, (W_s^n)_s$. According to the reflection principle and the Gaussian tail estimate (see e.g. Durrett (1991), Cpt 7 (3.8) and Cpt 1 (1.3))

$$\mathbb{P}\left(\sup_{0\le s\le t} W_s^1 > r\right) = 2\mathbb{P}\left(W_t^1 > r\right) \le \sqrt{\frac{2t}{r^2\pi}}\exp\left(-\frac{r^2}{2t}\right).$$

Now let us discuss continuity properties of harmonic maps and of solutions to the nonlinear Dirichlet problem. We firstly treat the question of (Lipschitz) continuity in the interior.

Corollary 6.11. *Assume that D is covered by open sets B with the property that for some $C, t > 0$ (depending on B)*

$$p^\diamond_{D,t}1 \le C\cdot\rho \quad \text{on } B\times B.$$

Then each bounded solution to the nonlinear Dirichlet problem on D is locally Lipschitz continuous on D.

The proof follows from Lemma 3.2(iv). For a typical example we refer to Example 4.8.

Definition 6.12. *(i) D is called* regular *if for each bounded measurable function $u : M \to \mathbb{R}$ the solution $v : M \to \mathbb{R}$ to the linear Dirichlet problem (in the sense of Definition 6.6 with $N = \mathbb{R}$) exists, is unique and is continuous in each point $z \in M\setminus D$ in which $u|_{M\setminus D}$ is continuous.*

(ii) D is called α-regular *(for some $\alpha \in]0,1]$) if it is regular and if there exists a constant C such that for each u and v from above*

$$\sup_{x\in M, y\in M\setminus D}\frac{|v(x)-v(y)|}{\rho(x,y)^\alpha} \le C\cdot \sup_{x,y\in M\setminus D}\frac{|u(x)-u(y)|}{\rho(x,y)^\alpha}.$$

In particular, if the semigroup $(p_t)_t$ is local then α-Hölder continuity of the boundary data u implies α-Hölder continuity of the solution v at the boundary.

Remark 6.13. A regular domain D is α-regular if and only if there exists a constant C and a symmetric function $\rho_* : M\times M \to [0,\infty]$ such that

- $p^\diamond_{D,t}\rho_* \le \rho_*$ on $M\times M$;
- $\rho_* \ge \rho^\alpha$ on $(M\setminus D)\times(M\setminus D)$;
- $\rho_* \le C\cdot\rho^\alpha$ on $(M\setminus D)\times D$.

Proof. Assume that D is α-regular for some fixed $\alpha \le 1$. For $z \in M\setminus D$ let $v_z(.)$ denote the solution to the linear Dirichlet problem for the function $x \mapsto u_z(x) := \rho(x,z)^\alpha$. Define a symmetric function ρ_* on $M\times M$ by $\rho_*(x,y) := \rho_0(x,y)\wedge\rho_0(y,x)$ where $\rho_0(x,y) := u_x(y)$ if $x\in M\setminus D, y\in M$ and $\rho_0(x,y) := +\infty$ else. By construction, $\rho_* = \rho^\alpha$ on $(M\setminus D)\times(M\setminus D)$ and, due to the assumption of α-regularity, $\rho_* \le C\cdot\rho^\alpha$ on $(M\setminus D)\times D$.

In order to prove $p^\diamond_{D,t}\rho_*(x,y) \le \rho_*(x,y)$ for given $x,y\in M$, we may assume without restriction that $\rho_*(x,y) = \rho_0(x,y) < \infty$. Hence, $x\in M\setminus D$ and thus $p_{D,t}(x,.) = \delta_x$. Therefore,

$$p^\diamond_{D,t}\rho_*(x,y) \quad = \quad \int \rho_*(x,z)\,p_{D,t}(y,dz) \ \le \ \int\rho_0(x,z)\,p_{D,t}(y,dz) \ = \ \rho_0(x,y) \ = \ \rho(x,y).$$

The reverse implication follows from Lemma 3.2(iv) (applied with $N = \mathbb{R}$), cf. also the proof of Theorem 6.15 below. □

There is a huge literature in analytic and probabilistic potential theory (classical as well as generalized) which deals with regular sets for the linear Dirichlet problem. Let us therefore restrict to mention the main example for α-regular domains.

Example 6.14. Let $M = \mathbb{R}^n$, let $(p_t)_t$ be the heat semigroup (= Brownian semigroup, Gaussian semigroup) and let D be a bounded domain in $\mathbb{R}^n$ which satisfies a uniform exterior cone condition with angle $\theta \in]0, \pi]$. Then there exists a number $\overline{\alpha} > 0$ (depending only on n and θ) such that D is α-regular for each $\alpha < \overline{\alpha}$.

For instance, $\overline{\alpha}(n, \pi) = 1$ for each $n \in \mathbb{N}$ (i.e. each convex domain is α-regular for each $\alpha < 1$) and $\overline{\alpha}(2, \theta) = (2 - \theta/\pi)^{-1}$ for each $\theta \in]0, \pi]$.

There exist no 1-regular domains $D \subset \mathbb{R}^n$. See Aikawa (2002).

Theorem 6.15. *(i) Assume that D is regular and (27) holds true. Then for each bounded map $f \in \mathcal{L}^*(M, N, p_D)$ the unique bounded map $g : M \to \mathbb{R}$ which solves the nonlinear Dirichlet problem (in the sense of Definition 6.6) is continuous in each point $z \in M \setminus D$ in which $f|_{M\setminus D}$ is continuous. In particular,*

$$\lim_{x \to z} g(x) = f(z).$$

(ii) Assume that D is α-regular for some $\alpha \in]0, 1]$ and (27) holds true. Let $f \in \mathcal{L}^*(M, N, p_D)$ be a bounded map such that $f|_{M\setminus D}$ is α-Hölder continuous. Then the solution g to the nonlinear Dirichlet problem is α-Hölder continuous in each point $z \in M \setminus D$. More precisely,

$$d(g(x), f(z)) \le C \cdot \rho(x, z)^\alpha \quad \text{for all } x \in M, z \in M \setminus D.$$

The constant C only depends on D, α and the Hölder norm of $f|_{M\setminus D}$.

(iii) If $(p_t)_t$ is local then in each of the above statements the set $M \setminus D$ may be replaced by ∂D.

Proof. (i) Fix $z \in M\setminus D$ and $f : M \to N$ such that the restriction of f to $M\setminus D$ is continuous in z. Define a function $u : M \to \mathbb{R}_+$ by $u(x) := d(f(x), f(z))$ and let v be the solution to the linear Dirichlet problem for u. Then by assumption v is continuous in z, $v(z) = 0$, v is harmonic and nonnegative on D and positive on $M \setminus D \setminus \{z\}$. (I.e. v is a barrier.) Now Lemma 3.2(iv) implies for all $x \in M$ and $t \in \mathbb{T}$

$$d(P^*_{D,t} f(x), P^*_{D,t} f(z)) \le p^\diamond_{D,t} d_f(x, z) = p_{D,t} u(x).$$

According to Corollary 6.8, for $t \to \infty$ the LHS converges to $d(g(x), g(z))$ and the RHS to $v(x)$. Hence, $d(g(x), g(z)) \le v(x)$ for all $x \in M$. This proves that g is continuous in z.

(ii), (iii) Slight modifications of the previous proof. □

For previous results on boundary continuity of generalized harmonic maps in more restrictive frameworks, we refer e.g. to Gregori (1998) and Fuglede (2002).

7 Harmonic maps characterized by convex and subharmonic functions

A complete characterization of harmonic maps in terms of convex and subharmonic functions is possible for some of the most important target spaces. Recall the definition of the operators A, A^* and $\underline{a}$ from Remark 6.5 and Definition 6.1, resp.

Proposition 7.1. *Let (N, d) be a simply connected, complete Riemannian manifold of nonpositive curvature. Let $f : M \to N$ be measurable with separable range and fix $x \in M$ such that for all $r > 0$*

$$\limsup_{t \to 0} \frac{1}{t} \int d^2(f(x), f(y))\, p_t(x, dy) < \infty \tag{29}$$

$$\lim_{t \to 0} \frac{1}{t} \int [d(f(x), f(y)) - r]_+ \, p_t(x, dy) = 0. \tag{30}$$

Then the following are equivalent:

(i) $Af(x) = 0$*;*

(ii) $\underline{a}(\varphi \circ f)(x) \geq 0$ *for all Lipschitz continuous, convex* $\varphi : N \to \mathbb{R}$*;*

and they are implied by

(iii) $f \in \mathcal{L}^*(M, N, p)$ *and* $A^* f(x) = 0$.

Remark 7.2. (i) Condition (29) and Lipschitz continuity of φ imply $\eta_t f(x) < \infty$ and $\eta_t(\varphi \circ f)(x) < \infty$ (at least for small t). Hence, $Af(x)$ and $\underline{a}(\varphi \circ f)(x)$ are well-defined.

(ii) Let ρ be a metric on M. Then conditions (29) and (30) are satisfied for all $f \in \mathrm{Lip}_\rho(M, N)$ provided for all $r > 0$

$$\limsup_{t \to 0} \frac{1}{t} \int \rho^2(x, y)\, p_t(x, dy) < \infty \tag{31}$$

$$\lim_{t \to 0} \frac{1}{t} \int \left[\rho(x, y) - r\right]_+ \, p_t(x, dy) = 0. \tag{32}$$

In terms of the Markov process associated with $(p_t)_t$, the first condition, is a linear bound for the quadratic variation, the second one a continuity condition. Under (31) it is equivalent to

$$\lim_{t \to 0} \frac{1}{t} p_t(x, M \setminus B_r(x)) = 0$$

$(\forall r > 0)$ which is a well known sufficient condition for continuity of paths of the stochastic process.

(iii) In assertion (ii) of the previous Proposition, one may restrict oneself to *smooth* functions φ. Indeed, in the following proof, one can easily smoothen out the functions φ_ζ.

Proof. According to Proposition 6.3 and Remark 6.5, it suffices to prove $(ii) \Rightarrow (i)$. Fix f and $x \in M$ and put $z_0 = f(x) \in N$. Our first aim is to construct convex functions φ on N which are almost linear around z_0. Recall that a smooth function φ on N is convex if and only if

$$\mathrm{Hess}\, \varphi(\xi, \xi) \geq 0 \qquad\qquad (\forall \xi \in SN).$$

For each $\zeta \in S_{z_0}N$ define a function $\tilde{\varphi}_\zeta : N \to \mathbb{R}$ by $\tilde{\varphi}_\zeta(w) = \langle \exp_{z_0}^{-1} w, \zeta \rangle$. Then $\mathrm{Hess}\, \tilde{\varphi}_\zeta(\xi, \xi) = 0$ for all $\zeta \in S_{z_0}N, \xi \in S_{z_0}N$. Hence, by continuity $\forall \varepsilon > 0 : \exists r > 0 : \forall z \in B_r(z_0), \forall \xi \in S_z N, \forall \zeta \in S_{z_0}N :$

$$|\mathrm{Hess}\, \tilde{\varphi}_\zeta(\xi, \xi)| \leq \varepsilon$$

Without restriction, we may assume $\varepsilon \leq 1$ and $r \leq 1$. On the other hand, we know that for $\psi := \frac{1}{2} d^2(., z_0)$

$$\mathrm{Hess}\, \psi(\xi, \xi) \geq 1 \qquad\qquad (\forall \xi \in SN).$$

Therefore, for each $\zeta \in S_{z_0}N$ the function $\hat{\varphi}_\zeta := \tilde{\varphi}_\zeta + \varepsilon \psi$ is convex on $B_r(z_0)$ and the function

$$\varphi_\zeta := \begin{cases} \sup \{\hat{\varphi}_\zeta, -r + 3d(z_0, .)\} & \text{on } B_r(z_0) \\ -r + 3d(z_0, .) & \text{on } N \setminus B_r(z_0) \end{cases}$$

is convex (and Lipschitz continuous) on N. The latter coincides with $\hat{\varphi}_\zeta$ on $B_{r/4}(z_0)$.

Now assume that $\underline{a}(\varphi \circ f)(x) \geq 0$ for each Lipschitz continuous convex $\varphi : N \to \mathbb{R}$. Then $\forall \varepsilon > 0 : \exists t_\varepsilon > 0 : \forall t \leq t_\varepsilon, \forall \zeta \in S_{z_0}N$:

$$\begin{aligned} 0 &= (\varphi_\zeta \circ f)(x) \ \leq \ p_t(\varphi_\zeta \circ f)(x) + \varepsilon t \ \stackrel{(*)}{\leq} \ p_t(\hat{\varphi}_\zeta \circ f)(x) + 2\varepsilon t \\ &= \int \langle \exp_{z_0}^{-1} f(y), \zeta \rangle p_t(x, dy) + \frac{\varepsilon}{2} \int d^2(z_0, f(y)) p_t(x, dy) + 2\varepsilon t. \end{aligned}$$

Here $(*)$ follows from (29) since

$$\begin{aligned} p_t(\varphi_\zeta \circ f)(x) - p_t(\hat{\varphi}_\zeta \circ f)(x) \ &= \ \int \left[\varphi_\zeta(f(y)) - \hat{\varphi}_\zeta(f(y))\right] p_t(x, dy) \\ \leq \ 4 \cdot \int \left[d(f(x), f(y)) - r/4\right]_+ \, p_t(x, dy) \ &\leq \ \varepsilon \cdot t. \end{aligned}$$

Recall (e.g. from Chavel (1993)) that $z_1 := P_t f(x)$ implies

$$\int \langle \exp_{z_1}^{-1} f(y), \zeta \rangle p_t(x, dy) = 0 \qquad\qquad (\forall \zeta \in S_{z_1}N).$$

Choose $\zeta_0 \in S_{z_0}N, \zeta_1 \in S_{z_1}N$ with $z_1 = \exp_{z_0}(d(z_0, z_1) \cdot \zeta_0), z_0 = \exp_{z_1}(d(z_0, z_1) \cdot \zeta_1)$. Then we may summarize

$$\begin{aligned} d(z_0, z_1) \leq \int & \left[-\langle \exp_{z_0}^{-1} f(y), \zeta_0 \rangle - \langle \exp_{z_1}^{-1} f(y), \zeta_1 \rangle + d(z_0, z_1)\right] p_t(x, dy) \\ & + \frac{\delta}{2} \cdot e_t f(x) + 2\varepsilon t \end{aligned} \tag{33}$$

with $e_t f(x) := \int d^2(f(x), f(y)) \, p_t(x, dy)$. In order to estimate the integrand in (33), consider an arbitrary triangle in a NPC space with side lengths a, b, c and angles α, β, γ. Then by triangle comparison

$$a^2 \geq b^2 + c^2 - 2bc \cos \alpha, \qquad b^2 \geq a^2 + c^2 - 2ac \cos \beta$$

and thus

$$0 \ \geq \ c - b \cdot \cos \alpha - a \cdot \cos \beta \tag{34}$$

Obviously, (33) and (34) together imply

$$d(z_0, z_1) \ \leq \ \frac{\delta}{2} \cdot e_t f(x) + 2\varepsilon t$$

But according to (30), $e_t f(x) \leq C \cdot t$ for $t \to 0$ and thus $\frac{1}{t} d(f(x), P_t f(x)) \to 0$ for $t \to 0$. That is, $Af(x) = 0$. $\square$

An even more complete picture is obtained if we require uniform convergence instead of pointwise convergence in the definitions of invariance and harmonicity. Hence, let us now consider harmonic maps in an $\mathcal{L}_\infty$-framework or, more generally, let us consider weakly harmonic maps in an L_θ-context. Recall that throughout this paper $(\delta_k)_k$ denotes a fixed subsequence of $(2^{-k})_k$.

Theorem 7.3. *Let $(M, \mathcal{M}, m)$ be a measure space and let $(p_t)_t$ be a Markov semigroup on $(M, \mathcal{M})$ satisfying the basic assumption (24) of Chapter 5, i.e. $\|p_t u\|_\theta \le C \cdot e^{\alpha t} \cdot \|u\|_\theta$ for some constants $C, \alpha \in \mathbb{R}$, $\theta \in [1, \infty]$ and all bounded measurable $u : M \to \mathbb{R}$.*

Let (N, d) be a simply connected, complete Riemannian manifold of nonpositive curvature. Finally, let $f : M \to N$ be a measurable map with separable range and such that for all $r > 0$

$$\limsup_{t\to 0} \frac{1}{t} \left[\int \left[\int d^2(f(x), f(y)) p_t(x, dy) \right]^\theta m(dx) \right]^{1/\theta} < \infty \tag{35}$$

$$\lim_{t\to 0} \frac{1}{t} \left[\int \left[\int [d(f(x), f(y)) - r]_+ \, p_t(x, dy) \right]^\theta m(dx) \right]^{1/\theta} = 0 \tag{36}$$

(with appropriate modifications if $\theta = \infty$). Then the following assertions are equivalent:

(i) *For each $t > 0$*

$$\overline{P}^*_t f := \lim_{\mathbb{T}\ni s\to 0} \overline{P}_s^{\lfloor t/s \rfloor} f$$

exists in $L_\theta(M, N, p)$ and $\overline{P}^*_t f = f$.

(ii) *There exists a subsequence $(s_k)_k$ of $(\delta_k)_k$ such that for all $t \in \mathbb{T}$ and for m-a.e. $x \in M$*

$$P^*_t f(x) := \lim_{k\to\infty} P_{s_k}^{\lfloor t/s_k \rfloor} f(x)$$

exists in N and $P^*_t f(x) = f(x)$.

(iii) *For each $t \in \mathbb{T}$, for m-a.e. $x \in M$ and for each convex, Lipschitz continuous $\varphi : N \to \mathbb{R}$*

$$(\varphi \circ f)(x) \le p_t(\varphi \circ f)(x).$$

(iv) $\lim_{\mathbb{T}\ni t\to 0} \frac{1}{t} d_\theta(f, \overline{P}_t f) = 0$.

(v) *For each $t \in \mathbb{T}$,*

$$\overline{P}^*_t f := \lim_{k\to\infty} \overline{P}_{\delta_k}^{\lfloor t/\delta_k \rfloor} f$$

exists in $L_\theta(M, N, p)$ and $\lim_{\mathbb{T}\ni t\to 0} \frac{1}{t} d_\theta(f, \overline{P}^*_t f) = 0$.

Proof. (i) $\Rightarrow$ (ii): Choose $t \in \mathbb{T}, s = \delta_k$ and recall that L_θ-convergence

$$\int \left[d\left(f, (P_{\delta_k})^{\lfloor t/\delta_k \rfloor} f\right)(x) \right]^\theta m(dx) \to 0 \qquad \text{for } k \to \infty$$

implies m-a.e. convergence for a suitable subsequence $(s_k)_k$ of $(\delta_k)_k$. Use a diagonal sequence argument to obtain convergence for all $t \in \mathbb{T}$.
(ii) $\Rightarrow$ (iii): By Jensen's inequality

$$\begin{aligned}(\varphi \circ f)(x) &= \varphi\left(P^*_t f\right)(x) = \lim_{k\to 0} \varphi\left(P_{s_k}^{\lfloor t/s_k \rfloor} f\right)(x) \\ &\le \lim_{k\to\infty} p_{s_k}^{\lfloor t/s_k \rfloor}(\varphi \circ f)(x) = p_t(\varphi \circ f)(x).\end{aligned}$$

(iii) $\Rightarrow$ (iv): Property (iii) and the estimates in the proof of Proposition 7.1 imply that for each $\delta > 0$ there exists $r > 0$ such that for m-a.e. $x \in M$ and all $t \in \mathbb{T}$

$$d(f(x), P_t f(x)) \le \delta \cdot e_t f(x) + 4 \int [d(f(x), f(y)) - r]_+ \, p_t(x, dy).$$

Hence, by (35) and (36)

$$\frac{1}{t} d_\theta(f, P_t f) \to 0$$

for $t \in \mathbb{T}, t \to 0$.
(iv) $\Rightarrow$ (i): According to (iv), $\lim_{\mathbb{T} \ni s \to 0} \frac{1}{s} d_\theta(f, P_s f) = 0$. Hence, for each $t > 0$

$$d_\theta(f, P_s^{\lfloor t/s \rfloor} f) \quad \leq \quad \sum_{i=1}^{\lfloor t/s \rfloor} d_\theta\left(P_s^{i-1} f, P_s^i f\right) \ \leq \ \lfloor t/s \rfloor \cdot d_\theta(f, P_s f) \to 0$$

as $s \in \mathbb{T}, s \to 0$. That is, for each $t > 0$

$$f = \lim_{\mathbb{T} \ni s \to 0} P_s^{\lfloor t/s \rfloor} f$$

in L_θ.
(i) $\Rightarrow$ (v): obvious.
(v) $\Rightarrow$ (i): Since $(P_t^*)_{t \in \mathbb{T}}$ is a semigroup, it follows that for all $t \in \mathbb{T}$ and $n \in \mathbb{N}$

$$d_\theta(f, P_t^* f) = d_\theta\left(f, \left(P_{t/2^n}^*\right)^{2^n} f\right) \leq 2^n \cdot d_\theta\left(f, P_{t/2^n}^* f\right) \to 0$$

as $n \to \infty$.

$\square$

Remark 7.4. (i) Assumption (35) guarantees that $f \in L_\theta(M, N, p)$. Thus $\overline{P}_t f$ is well defined. Moreover, it is strongly continuous. Hence, most results easily extend from $s, t \in \mathbb{T}$ to $s, t \in \mathbb{R}_+$.

(ii) In the framework of the previous Theorem, the notion of L_θ-harmonic maps will be independent of the choice of the sequence $(\delta_k)_k$.

(iii) In the above situation, a map is L_θ-invariant if and only if it is L_θ-harmonic (which in turn holds if and only if it is L_θ-pseudo harmonic).

(iv) If $\theta = \infty$ and if m is the counting measure, then in assertion (ii) of the above Theorem one may choose any null sequence $(s_k)_k$.

The previous characterization is analogous to Ishihara's characterization of classical harmonic maps (Ishihara (1979)). As a consequence of the previous Theorem we immediately obtain the following

Corollary 7.5. *Let M and N be Riemannian manifolds, let $(p_t)_t$ be the heat semigroup on M, assume that M is complete with lower bounded Ricci curvature and that N is complete, simply connected and nonpositively curved. Then for any bounded, continuous map $f : M \to N$ and any open set $D \subset M$ the following assertions are equivalent:*

(i) *$Af(x) = 0$ for all $x \in D$;*

(ii) *$A^* f(x) = 0$ for all $x \in D$;*

(iii) *For all open sets B which are relatively compact in D, all $x \in M$ and all $t > 0$*

$$P_{B,t}^* f(x) = f(x);$$

(iv) *For all open sets B which are relatively compact in D, all $x \in M$, all $t > 0$ and all Lipschitz continuous, convex $\varphi : N \to \mathbb{R}$:*

$$(\varphi \circ f)(x) \leq p_{B,t}(\varphi \circ f)(x);$$

(v) *For all $t > 0$, all $x \in D$ and all Lipschitz continuous, convex φ:*

$$\underline{a}(\varphi \circ f)(x) \geq 0;$$

(vi) *f is harmonic on D in the classical sense.*

Moreover, in (iii)-(v) one may restrict oneself to smooth functions φ and balls B.

Proof. The assumptions on M, N and f guarantee that $(p_t)_t$ is a Markov semigroup and that $f \in \mathcal{L}^*(M, N, p)$ (for a suitably chosen sequence $(\delta_k)_k$, cf. Theorem 4.3).
(i) $\Rightarrow$ (v) as well as (ii) $\Rightarrow$ (v): Proposition 6.3.
(v) $\Rightarrow$ (iv): Remark 6.2 applied to $u := \varphi \circ f$.
(iv) $\Rightarrow$ (iii): Apply the previous Theorem 7.3 (with uniform norm, i.e. $\theta = \infty$ and m= counting measure) to the semigroup $(p_{B,t})_t$.
(iii) $\Rightarrow$ (i) and (ii): The previous Theorem 7.3 states that (iii) implies
(i') $\lim\limits_{t\to 0} \frac{1}{t} d(f, P_{B,t}f)(x) = 0$ uniformly in $x \in M$;
as well as
(ii') $f \in \mathcal{L}^*(M, N, p_B)$ and $\lim\limits_{t\to 0} \frac{1}{t} d(f, P^*_{B,t}f)(x) = 0$ uniformly in $x \in M$.
According to Proposition 6.9: (ii') $\Rightarrow$ (ii) and (with the same argument) (i') $\Rightarrow$ (i);
(v) $\Leftrightarrow$ (vi): Ishihara (1979) and Remark 6.2. $\square$

Similar characterizations of harmonic maps in terms of subharmonic functions and convex functions may be obtained for target spaces more general than Riemannian manifolds. We state one result for metric trees (with general domain spaces) and one results for Riemannian polyhedra (with Riemannian domain spaces).

Proposition 7.6. *Let (N, d) be a locally finite metric tree, let $x \in M$ and $f : M \to N$ be measurable and satisfying property (30). Then the following are equivalent:*

(i) $Af(x) = 0$*;*

(ii) $\underline{a}(\varphi \circ f)(x) \geq 0$ *for all Lipschitz continuous, convex $\varphi : N \to \mathbb{R}$.*

Before coming to the proof of this Proposition, let us derive some auxiliary results and introduce some notations. Given a metric tree N and a point $z \in N$ we denote by T_zN the equivalence class of unit speed geodesics emanating from z where two such geodesics are called equivalent if they coincide on some open interval. Given $\gamma \in T_zN$ we define the *oriented distance* $d_\gamma : N \to \mathbb{R}$ by $d_\gamma(y) := d(z, y)$ if the geodesic connecting z and y is equivalent to γ and $d_\gamma(y) := -d(z, y)$ otherwise.

Lemma 7.7. *Let $q \in \mathcal{P}^1(N)$, $z \in N$ and $r > 0$.*
(i) $z = b(q) \quad \Longleftrightarrow \quad \forall \gamma \in T_zN : \int d_\gamma(y)\, q(dy) \leq 0.$
(ii) If $\varphi(z) \leq \int \varphi(y)\, q(dy)$ for all Lipschitz continuous convex $\varphi : N \to \mathbb{R}$ and if there is no branch point (besides eventually z) in the ball $B_r(z)$ then

$$d(z, b(q)) \leq \int [d(z, y) - r]_+ \, q(dy).$$

Proof. (i) According to the basic barycenter contraction property, we may assume that $q \in \mathcal{P}^2(N)$. Then by definition of the L_2-barycenter

$$z = b(q) \qquad \Longleftrightarrow \qquad \forall \gamma \in T_zN : \frac{d}{dt} \int d^2(\gamma_t, y)\, q(dy)|_{t=0+} \geq 0.$$

However, it is easy to see that $\frac{d}{dt}d^2(\gamma_t, y)|_{t=0+} = -2d_\gamma(y)$. This proves the claim.

(ii) Let $\gamma \in T_{b(q)}N$ be the geodesic connecting $b(q)$ and z. Consider the following truncation of the oriented distance

$$\varphi(.) := \sup\{d_\gamma(.), d(z, b(q)) - r\}.$$

Then φ is convex, dil$\varphi \leq 1$ and thus by assumption

$$d(z, b(q)) = \varphi(z) \leq \int \varphi(y)\, q(dy) \leq \int d_\gamma(y)\, q(dy) + \int [d(z, y) - r]_+\, q(dy) \leq \int [d(z, y) - r]_+\, q(dy)$$

where the last inequality follows from (i). □

Actually, the above proof shows that it suffices to verify the assumption $\varphi(z) \leq \int \varphi(y)\, q(dy)$ for all $\varphi = (d_\eta)_+$ with $w \in \partial B_r(z)$ and $\eta \in T_w N$ being the geodesic connecting w and z.

More generally, $\varphi(z) \leq \int \varphi(y)\, q(dy) + \beta$ for all such φ and some number $\beta \in \mathbb{R}$ implies that $d(z, b(q)) \leq \int [d(z, y) - r]_+\, q(dy) + \beta$.

Proof of Proposition 7.6 It suffices to prove (ii) $\Rightarrow$ (i). Choose $r > 0$ such that $B_r(f(x)) \setminus \{f(x)\}$ contains no branch points and let $\Phi_x := \{\varphi_w : w \in \partial B_r(f(x))\}$ denote the set of Lipschitz continuous, convex functions $\varphi = (d_\eta)_+$ with $\eta \in T_w N$ being the geodesic connecting some $w \in \partial B_r(f(x))$ and $f(x)$. Since $f(x)$ has finite degree, this is a finite set. Assumption (ii) implies $\underline{a}(\varphi \circ f)(x) \geq 0$ for all $\varphi \in \Phi_x$. That is,

$$\forall \varepsilon > 0 : \exists t_\varepsilon > 0 : \forall t \leq t_\varepsilon, \forall \varphi \in \Phi_x : (\varphi \circ f)(x) \leq p_t(\varphi \circ f)(x) + \varepsilon t$$

According to Lemma 7.7 and the subsequent remarks (with $z = f(x), q = f_* p_t(x, .)$ and $b(q) = P_t f(x)$) this yields

$$d\left(f(x), P_t f(x)\right) \quad \leq \quad \varepsilon t + \int [d\left(f(x), f(y)\right) - r]_+\, p_t(x, dy).$$

Hence, (30) implies $\frac{1}{t} d(f(x), P_t f(x)) \to 0$ for $t \to 0$. □

Corollary 7.8. *Let M be a complete Riemannian manifold with lower bounded Ricci curvature and let $(p_t)_t$ be the heat semigroup on M. Fix a countable basis $\mathfrak{B}_0$ of the topology of M consisting of balls. Let N be a complete, simply connected and nonpositively curved Riemannian polyhedron of dimension $n \leq 2$. Then for any bounded, continuous map $f : M \to N$ and any open set $D \subset M$ the following assertions are equivalent:*

(i) $A^* f(x) = 0$ *for all* $x \in D$;

(ii) *For all balls $B \in \mathfrak{B}_0$ which are relatively compact in D, all $x \in M$ and all $t > 0$:*

$$P^*_{B,t} f(x) = f(x);$$

(iii) *For all balls B which are relatively compact in D, all $x \in M$, all $t > 0$ and all Lipschitz continuous, convex $\varphi : N \to \mathbb{R}$:*

$$(\varphi \circ f)(x) \leq p_{B,t}(\varphi \circ f)(x);$$

(iv) *For all $t > 0$, all $x \in D$ and all Lipschitz continuous, convex φ:*

$$\underline{a}(\varphi \circ f)(x) \geq 0;$$

(iv) *f is harmonic on D in the sense of* EELLS, FUGLEDE *(2001).*

Moreover, if $n = 1$ (i.e. if N is a tree) then the above assertions are also equivalent to

(vi) *$Af(x) = 0$ for all $x \in D$.*

Proof. The assumptions on M, N and f guarantee that $(p_t)_t$ is a Markov semigroup and that $f \in \mathcal{L}^*(M, N, p) \cap \bigcap_{B \in \mathfrak{B}_0} \mathcal{L}^*(M, N, p_B)$ (for a suitably chosen sequence $(\delta_k)_k$, cf. Theorem 4.3).
(ii)⇒(i): Proposition 6.9.
(i)⇒(iv) as well as (vi)⇒(iv): Proposition 6.3 (Jensen's inequality).
(iv)⇒(iii): Remark 6.2 (classical potential theory).
(iii)⇒(v): FUGLEDE (2002).
(v)⇒(ii): Fix a map f satisfying (v) and a ball B which is relatively compact in D. Let $g := \lim_{t\to\infty} P^*_{B,t} f$ be the solution to the nonlinear Dirichlet problem on B (for the boundary data f) as defined in the previous Chapter. Then $g = P^*_{B,t} g$ in B for all $t > 0$ and thus $A^*_B g = 0$ on B. Moreover, g is continuous on B (Corollary 6.11) as well as on ∂B (Theorem 6.15) and it coincides with f on $M \setminus B$. Hence, (by the previous arguments) g is harmonic on B in the sense of EELLS, FUGLEDE (2001) and therefore, (by the uniqueness of the solution to the Dirichlet problem) it coincides with f on the whole space M. Therefore, $f = P^*_{B,t} f$ in B for all $t > 0$.
(iv)⇒(vi): Proposition 7.6. □

8 Reverse Variance Inequality and Convergence

In this Chapter, we study maps into global NPC spaces (N, d) with some additional weak bound for the "curvature" which will be expressed in terms of a so-called reverse variance inequality. We recall from STURM (2001) that "nonpositive curvature" can be characterized in terms of the "variance inequality".

Proposition 8.1. *A complete metric space (N, d) has globally nonpositive curvature (in the sense of Alexandrov) if and only if for each $q \in \mathcal{P}^2(N)$ there exists a (unique) point $b(q) \in N$ such that $\forall z \in N$*

$$\int \left[d^2(z, x) - d^2(z, b(q)) - d^2(b(q), x)\right] q(dx) \geq 0.$$

Spaces with this property are called global NPC spaces. Note that a simple application of the triangle inequality yields

$$\int \left[d^2(z, x) - d^2(z, b(q)) - d^2(b(q), x)\right] q(dx) \leq \int \left[d^2(z, b(q)) + d^2(b(q), x)\right] q(dx).$$

The crucial point in the reverse variance inequality which we will formulate below is that for $d \to 0$ the RHS is of order d^α for some $\alpha > 2$.

Definition 8.2. We say that a *reverse variance inequality with exponent* $\alpha > 2$ holds true on a global NPC space (N, d) iff there exists a constant c such that

$$\int \left[d^2(z, x) - d^2(z, b(q)) - d^2(b(q), x)\right] q(dx) \leq c \cdot \int \left[d^\alpha(z, b(q)) + d^\alpha(b(q), x)\right] q(dx)$$

for all $q \in \mathcal{P}^2(N)$ and all $z \in N$.

Our next goal is to prove that a reverse variance inequality with exponent $\alpha = 4$ holds true on each global NPC space with lower bounded curvature in the sense of Alexandrov. In particular, it therefore will hold on each simply connected, complete Riemannian manifold with lower bounded and nonpositive curvature. Trivially, it also holds on each Hilbert space. We leave it as an exercise for the reader to verify that it also holds on singular spaces like the following

Example 8.3. Glue together two copies of the set $\{z = (x,t) \in \mathbb{R}^k : t \leq \psi(x)\}$ along their boundary $\{z = (x,t) \in \mathbb{R}^k : t = \psi(x)\}$ where $\psi : \mathbb{R}^{k-1} \to \mathbb{R}$ is any smooth convex function.

For the following calculations, put $\mathrm{sh}_\kappa r := \frac{1}{\kappa} \cdot \sinh(\kappa \cdot r)$, $\mathrm{ch}_\kappa r := \cosh(\kappa \cdot r)$ for $\kappa > 0$ and $\mathrm{sh}_0 r = r$, $\mathrm{ch}_0 r = 1$.

Lemma 8.4. *Assume that (N, d) is a geodesically complete, global NPC space with curvature $\geq -\kappa^2$. Then for each $q \in \mathcal{P}^2(N)$ and each $z \in N$*

$$\int \mathrm{ch}_\kappa d(z,x) q_\kappa(dx) \leq \mathrm{ch}_\kappa d(z, b(q)) \cdot \int \mathrm{ch}_\kappa d(b(q), x) q_\kappa(dx) \tag{37}$$

where $q_\kappa(dx) := \frac{d(x,b(q))}{\mathrm{sh}_\kappa d(x,b(q))} q(dx)$ if $\kappa > 0$. In the limit case $\kappa = 0$, (37) should be replaced by the variance equality $\int d^2(z,x) q(dx) = d^2(z, b(q)) + \int d^2(b(q), x) q(dx)$.

Proof. Let $\kappa > 0$ and fix a probability measure q and a point z. Consider the geodesic connecting $b(q)$ and z. By geodesical completeness, it can be extended beyond $b(q)$. That is, for $t > 0$ sufficiently small, there exists a geodesic $\gamma : [-t, 1] \to N$ with $\gamma_0 = b(q)$ and $\gamma_1 = z$. The lower curvature bound implies

$$\mathrm{sh}_\kappa \left[(1+t) d(z, b(q))\right] \cdot \mathrm{ch}_\kappa d(x, b(q)) \geq \mathrm{sh}_\kappa d(z, b(q)) \cdot ch_\kappa d(x, \gamma_{-t}) + \mathrm{sh}_\kappa \left[t \cdot d(z, b(q))\right] \cdot \mathrm{ch}_\kappa d(x, z)$$

for all $x \in N$ (cf. Korevaar, Schoen (1997)). Integrating with respect to q_κ yields

$$\begin{aligned}
\int \mathrm{ch}_\kappa d(x,z) q_\kappa(dx) \quad &\leq \quad \frac{\mathrm{sh}_\kappa \left[(1+t) d(z,b(q))\right] - \mathrm{sh}_\kappa d(z,b(q))}{\mathrm{sh}_\kappa \left[t d(z,b(q))\right]} \cdot \int \mathrm{ch}_\kappa d(x, b(q)) q_\kappa(dx) \\
&+ \quad \frac{\mathrm{sh}_\kappa d(z,b(q))}{\mathrm{sh}_\kappa \left[t d(z,b(q))\right]} \cdot \int \left[\mathrm{ch}_\kappa d(x,b(q)) - \mathrm{ch}_\kappa d(x, \gamma_{-t})\right] q_\kappa(dx)
\end{aligned}$$

for all sufficiently small t and thus in the limit $t \searrow 0$

$$\begin{aligned}
& \int \mathrm{ch}_\kappa d(x,z) q_\kappa(dx) - \mathrm{ch}_\kappa d(z, b(q)) \cdot \int \mathrm{ch}_\kappa d(x, b(q)) q_\kappa(dx) \\
\leq \quad & \frac{\mathrm{sh}_\kappa d(z,b(q))}{d(z,b(q))} \cdot \liminf_{t \searrow 0} \frac{1}{t} \int \left[\mathrm{ch}_\kappa d(x, \gamma_0) - \mathrm{ch}_\kappa d(x, \gamma_{-t})\right] q_\kappa(dx) \\
\overset{(+)}{\leq} \quad & \frac{\kappa^2 \mathrm{sh}_\kappa d(z,b(q))}{2 d(z,b(q))} \cdot \liminf_{t \searrow 0} \frac{1}{t} \int \left[d^2(x,\gamma_0) - d^2(x, \gamma_{-t})\right] \frac{\mathrm{sh}_\kappa d(x,\gamma_0)}{d(x,\gamma_0)} q_\kappa(dx) \\
= \quad & \frac{\kappa^2 \mathrm{sh}_\kappa d(z,b(q))}{2 d(z,b(q))} \cdot \liminf_{t \searrow 0} \frac{1}{t} \int \left[d^2(x,\gamma_0) - d^2(x, \gamma_{-t})\right] q(dx) \\
\overset{(++)}{\leq} \quad & 0
\end{aligned}$$

where $(+)$ is due to the fact that for all $R_0, R_t > 0$

$$\frac{1}{t}\left[\mathrm{ch}_\kappa R_0 - \mathrm{ch}_\kappa R_t\right] = \frac{\mathrm{ch}_\kappa R_0 - \mathrm{ch}_\kappa R_t}{R_0^2 - R_t^2} \cdot \frac{R_0^2 - R_t^2}{t} \le \frac{\kappa^2 \cdot \mathrm{sh}_\kappa R_0}{2R_0} \cdot \frac{R_0^2 - R_t^2}{t}$$

and $(++)$ due to the fact that $\gamma_0 = b(q)$ is the barycenter of q. The case $\kappa = 0$ follows analogously. □

Theorem 8.5. *On each geodesically complete, global NPC space (N,d) with curvature $\ge -\kappa^2$ a reverse variance inequality with exponent 4 and constant $\frac{2}{3}\kappa^2$ holds true. That is, for each $q \in \mathcal{P}^2(N)$ and for each $z \in N$*

$$\int \left[d^2(z,x) - d^2(z,b(q)) - d^2(b(q),x)\right] q(dx) \quad \le \quad \frac{2}{3}\kappa^2 \int \left[d^4(z,b(q)) + d^4(b(q),x)\right] q(dx).$$

Proof. Put $D = d(z,x), d_1 = d(x,b(q)), d_2 = d(b(q),z), d = \frac{d_1+d_2}{2}$ and assume for simplicity $\kappa = 1$. Then

$$\begin{aligned}
\int \left[d^2(z,x) - d^2(z,b(q)) - d^2(b(q),x)\right] q(dx) \quad &= \quad \int \left[D^2 - d_1^2 - d_2^2\right] dq \\
= \quad & \int \left[D^2\left(1 - \frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2}\right) + D^2 \cdot \frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2} - d_1^2 - d_2^2\right] dq \\
\le \quad & \int \left[D^2\left(1 - \frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2}\right) + 2\left(\frac{1}{2}D^2 - \mathrm{ch}D + \mathrm{ch}d_1\cdot\mathrm{ch}d_2\right)\cdot\frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2} - d_1^2 - d_2^2\right] dq \\
\le \quad & \int \left[(d_1+d_2)^2\left(1 - \frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2}\right) + 2(\mathrm{ch}d_1\cdot\mathrm{ch}d_2 - 1)\cdot\frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2} - d_1^2 - d_2^2\right] dq \\
= \quad & 2\int \left[\mathrm{ch}(2d) - 1 - 2d^2\right] \frac{d_1}{\mathrm{sh}d_1}\cdot\frac{d_2}{\mathrm{sh}d_2} dq \\
\overset{(*)}{\le} \quad & 2\int \left[\mathrm{ch}(2d) - 1 - 2d^2\right]\cdot\left(\frac{d}{\mathrm{sh}d}\right)^2 dq \quad = \quad 4\int\left[1 - \left(\frac{d}{\mathrm{sh}d}\right)^2\right]\cdot d^2 dq \\
\le \quad & \frac{4}{3}\int d^4 dq \quad \le \quad \frac{2}{3}\int\left[d_1^4 + d_2^4\right] dq \quad = \quad \frac{2}{3}\int\left[d^4(x,b(q)) + d^4(b(q),z)\right] q(dx).
\end{aligned}$$

The inequality (*) follows from the logarithmic concavity of the function $r \mapsto \frac{r}{\mathrm{sh}r}$. □

Remark 8.6. Consider the convex, increasing function $\phi_\kappa : r \mapsto \left(1 - (\mathrm{sh}_\kappa r/r)^2\right)\cdot r^2$ on $\mathbb{R}_+$ which satisfies $\phi_\kappa(r) \le \frac{\kappa^2}{3}r^4$ as well as $\phi_\kappa(r) \le r^2$. The previous proof yields the sharper estimate

$$\int \left[D^2 - d - 1^2 - d_2^2\right] dq \le 4\int \phi_\kappa(d)dq \le 2\int \left[\phi_\kappa(d_1) + \phi_\kappa(d_2)\right] dq.$$

Now let us turn to the proof of Theorem 4.1. Firstly, we will formulate some auxiliary results.

Definition 8.7. For $f \in \mathcal{L}(M,N,p), n \in \mathbb{N}$ and $\alpha > 0$ we define the *α-order iterated variation*

$$\begin{aligned}
v_{t,n}^{(\alpha)} f(x) \quad = \quad & \int \ldots \int \Big[d^\alpha\left(P_{t/n}^n f(x), P_{t/n}^{n-1} f(x_1)\right) + d^\alpha\left(P_{t/n}^{n-1} f(x_1), P_{t/n}^{n-2} f(x_2)\right) + \ldots \\
& \ldots + d^\alpha\left(P_{t/n} f(x_{n-1}), f(x_n)\right)\Big]\, p_{\frac{t}{n}}(x_{n-1}, dx_n) \ldots p_{\frac{t}{n}}(x, dx_1)
\end{aligned}$$

and the deviation

$$\delta_{t,n} f(x) = \int d^2\left(P^n_{t/n} f(x), f(y)\right) p_t(x,dy) - v^{(2)}_{t,n} f(x).$$

Note that $\delta_{t,1} f(x) \equiv 0$ and that $\delta_{t,n} f(x) \geq d^2\left(P^n_{t/n} f(x), P_t f(x)\right)$ by the following Lemma.

Lemma 8.8. *For all $f, g \in \mathcal{L}(M,N,p)$, all $t > 0$ and all $k, n \in \mathbb{N}$*

$$d^2\left(P^k_{t/k} f(x), P^n_{t/n} g(x)\right) \leq \left[\int d(f(y), g(y)) p_t(x,dy)\right]^2 + \delta_{t,k} f(x) + \delta_{t,n} g(x)$$

and

$$d^2\left(P^{kn}_{t/(kn)} f(x), P^n_{t/n} f(x)\right) \leq \sum_{i=0}^{n-1} p_{\frac{n-1-i}{n}t}\left(\delta_{\frac{t}{n},k}\left(P^{ki}_{t/(kn)} f\right)\right)(x).$$

Proof. Iterated application of the variance inequality yields

$$\begin{aligned}
& P^k_{t/k} f(x), h(x) \\
\leq \quad & \int d^2\left(h(x), P^{k-1}_{t/k} f(x_1)\right) p_{\frac{t}{k}}(x,dx_1) - \int d^2\left(P^k_{t/k} f(x), P^{k-1}_{t/k} f(x_1)\right) p_{\frac{t}{k}}(x,dx_1) \\
\leq \quad & \int\int d^2\left(h(x), P^{k-2}_{t/k} f(x_2)\right) p_{\frac{t}{k}}(x_1,dx_2) p_{\frac{t}{k}}(x,dx_1) \\
& - \int\int d^2\left(P^{k-1}_{t/k} f(x_1), P^{k-2}_{t/k} f(x_2)\right) p_{\frac{t}{k}}(x_1,dx_2) p_{\frac{t}{k}}(x,dx_1) \\
& - \int d^2\left(P^k_{t/k} f(x), P^{k-1}_{t/k} f(x_1)\right) p_{\frac{t}{k}}(x,dx_1) \\
\leq \ldots \leq \quad & \int d^2(h(x), f(y)) p_t(x,dy) - v^{(2)}_{t,k} f(x)
\end{aligned}$$

for all h. Choosing $h = P^n_{t/n} g$ yields

$$d^2\left(P^k_{t/k} f(x), P^n_{t/n} g(x)\right) \leq \int d^2\left(P^n_{t/n} g(x), f(y)\right) p_t(x,dy) - v^{(2)}_{t,k} f(x).$$

Interchanging the roles of f, g and k, n we obtain similarly

$$d^2\left(P^k_{t/k} f(x), P^n_{t/n} g(x)\right) \leq \int d^2\left(P^k_{t/k} f(x), g(y)\right) p_t(x,dy) - v^{(2)}_{t,n} g(x).$$

Adding up both inequalities and applying the quadruple inequality (see e.g. Korevaar, Schoen (1993)) to $P^k_{t/k} f(x)$, $P^n_{t/n} g(x)$, $g(y)$, $f(y)$ we obtain

$$\begin{aligned}
& d^2\left(P^k_{t/k} f(x), P^n_{t/n} g(x)\right) \\
\leq \quad & \left[\int d(f(y), g(y)) p_t(x,dy)\right]^2 + \int d^2\left(P^k_{t/k} f(x), f(y)\right) p_t(x,dy) \\
& + \int d^2\left(P^n_{t/n} g(x), g(y)\right) p_t(x,dy) - v^{(2)}_{t,k} f(x) - v^{(2)}_{t,n} g(x) \\
= \quad & (p_t u(x))^2 + \delta_{t,k} f(x) + \delta_{t,n} g(x)
\end{aligned}$$

which is the first claim.

For $i = 0, .., n-1$ the preceding yields

$$d^2\left(P^{k(i+1)}_{t/(kn)}f, P^{i+1}_{t/n}f\right)(x) \le p_{\frac{t}{n}}\left(d^2\left(P^{ki}_{t/(kn)}f, P^{i}_{t/n}f\right)\right)(x) + \delta_{\frac{t}{n},k}\left(P^{ki}_{t/(kn)}f\right)(x).$$

Iterating this inequality proves the second claim. □

Proposition 8.9. *Assume that a reverse variance inequality with exponent $\alpha > 2$ holds true on (N, d).*

(i) Then the pointwise limit

$$P^*_t f(x) := \lim_{k\to\infty} P^{2^k}_{t/2^k} f(x) \tag{38}$$

exists for all $f \in \mathcal{L}(M, N, p)$, all $t > 0$ and all $x \in M$ with

$$\sum_{k=1}^{\infty} v^{(\alpha)}_{t,2^k} f(x) < \infty. \tag{39}$$

(ii) If even

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} v^{(\alpha)}_{t,(2^k n)} f(x) = 0 \tag{40}$$

then

$$P^*_t f(x) = \lim_{n\to\infty} P^n_{t/n} f(x). \tag{41}$$

Proof. (i) According to the reverse variance inequality

$$\begin{aligned}
&\delta_{\frac{t}{ni^2}}(f_i)(x) := \\
&\qquad \int\int \left[d^2(P^2_{t/(2n)}f_i(x), f_i(z)) - d^2(P^2_{t/(2n)}f_i(x), P_{t/(2n)}f_i(y)) - d^2(P_{t/(2n)}f_i(y), f_i(z))\right] \\
&\qquad\qquad p_{\frac{t}{2n}}(y, dz)p_{\frac{t}{2n}}(x, dy) \\
&\le \; c \cdot \int\int \left[d^{2+\varepsilon}\left(f_{i+2}(x), f_{i+1}(y)\right) + d^{2+\varepsilon}\left(f_{i+1}(y), f_i(x)\right)\right] p_{\frac{t}{2n}}(y, dz)p_{\frac{y}{2n}}(x, dy) \\
&\le \; 2c \cdot \left(\frac{t}{2n}\right)^{1+\delta}
\end{aligned}$$

for $f_i := \left(P_{t/(2n)}\right)^i f, i = 0, 1, ..., 2n$. Therefore, by Lemma 8.8

$$d^2\left(P^n_{t/n}f, P^{2n}_{t/(2n)}f\right)(x) \le \sum_{i=0}^{n-1} p_{\frac{n-1-i}{n}t}\left(\delta_{\frac{t}{n},2}(f_{2i})\right)(x) \le c \cdot v^{(\alpha)}_{t,2n} f(x).$$

Iterating this inequality yields

$$d^2\left(P^n_{t/n}f, P^{2^k n}_{t/(2^k n)}f\right)(x) \le c \cdot \sum_{i=1}^{k} v^{(\alpha)}_{t,2^i n} f(x). \tag{42}$$

This together with condition (39) implies that $\left(P^{2^k}_{t/2^k}f(x)\right)_{k\in\mathbb{N}}$ is a Cauchy sequence. Hence, $P^*_t f(x) = \lim_{k\to\infty} P^{2^k}_{t/2^k} f(x)$ exists in N.

(ii) Our next aim is to prove that $P^*_t f(x) = \lim_{n\to\infty} P^n_{t/n} f(x)$. But

$$d\left(P^n_{t/n}f, P^*_t f\right)(x) \leq d\left(P^n_{t/n}f, P^{2^k n}_{t/(2^k n)}f\right)(x) + d\left(P^{2^k n}_{t/(2^k n)}f, P^{2^k}_{t/(2^k)}f\right)(x) + d\left(P^{2^k}_{t/(2^k)}f, P^*_t f\right)(x)$$

and, according to (42) and condition (40)

$$d^2\left(P^n_{t/n}f, P^{2^k n}_{t/(2^k n)}f\right)(x) \leq c \cdot \sum_{i=1}^{\infty} v^{(\alpha)}_{t,2^i n} f(x) \to 0$$

as $n \to \infty$ (uniformly in k) whereas, again according to (42),

$$d^2\left(P^{2^k}_{t/(2^k)}f, P^*_t f\right)(x) \leq c \cdot \sum_{i=k+1}^{\infty} v^{(\alpha)}_{t,2^i} f(x) \to 0$$

as $k \to \infty$ (uniformly in n). Using Lemma 8.8 and the reverse variance inequality, we can bound the remaining term as follows

$$d^2\left(P^{2^k n}_{t/(2^k n)}f, P^{2^k}_{t/(2^k)}f\right)(x) \leq \sum_{i=0}^{2^k-1} p_{\frac{2^k-1-i}{2^k}t}\left(\delta_{\frac{t}{2^k},n}(f_{ni})\right)(x)$$

with $f_i = P^i_s f$ and $s = \frac{t}{2^k n}$. Now by the reverse variance inequality

$$\begin{aligned}
&\delta_{sn,n}(f_i)(x_0) = \\
&= \int \dots \int \left[d^2\left(P^n_s f_i(x_0), f_i(x_n)\right) - \sum_{l=0}^{n-1} d^2\left(P^{n-l}_s f_i(x_l), P^{n-l-1}_s f_i(x_{l+1})\right)\right] \\
&\qquad p_s(x_{n-1}, dx_n) \dots p_s(x_0, dx_1) \\
&= \sum_{l=1}^{n-1} \int \dots \int [d^2\left(P^n_s f_i(x_0), P^{n-l-1}_s f_i(x_{l+1})\right) - d^2\left(P^n_s f_i(x_0), P^{n-l}_s f_i(x_l)\right) - \\
&\quad - d^2\left(P^{n-l}_s f_i(x_l), P^{n-l-1}_s f_i(x_{l+1})\right)] p_s(x_{n-1}, dx_n) \dots p_s(x_0, dx_1) \\
&\leq c \cdot \sum_{l=1}^{n-1} \int \dots \int \left[d^\alpha\left(P^n_s f_i(x_0), P^{n-l}_s f_i(x_l)\right) + d^\alpha\left(P^{n-l}_s f_i(x_l), P^{n-l-1}_s f_i(x_{l+1})\right)\right] \\
&\qquad p_s(x_{n-1}, dx_n) \dots p_s(x_0, dx_1) \\
&\leq c \cdot n^{\alpha-1} \cdot \sum_{l=0}^{n-1} \int \dots \int d^\alpha\left(P^{n-l}_s f_i(x_l), P^{n-l-1}_s f_i(x_{l+1})\right) p_s(x_{n-1}, dx_n) \dots p_s(x_0, dx_1).
\end{aligned}$$

Thus

$$\begin{aligned}
&d^2\left(P^{2^k n}_{t/(2^k n)}f, P^{2^k}_{t/(2^k)}f\right)(x) \\
&\leq c \cdot n^{\alpha-1} \cdot \sum_{i=0}^{2^k-1} \sum_{l=0}^{n-1} d^\alpha\left(f_{n-l+in}(y), f_{n-l+in-1}(z)\right) p_s(y, dz) p_{(2^k-1-i)ns+ls}(x, dy) \\
&= c \cdot n^{\alpha-1} \cdot v^{(\alpha)}_{t,2^k n} f(x)
\end{aligned}$$

which for each $n \in \mathbb{N}$ is arbitrarily small if k is sufficiently large. Hence, $d\left(P_{t/n}^n f, P_t^* f\right)(x) \to 0$ for $n \to \infty$. $\square$

Theorem 8.10. *Assume that for suitable constants $\alpha > 2, \beta > 0$ and C and a given metric ρ on M*

- $p_t^\diamond \rho(x,y) \le C \cdot \rho(x,y) \qquad (\forall x, y \in M, t \le 1)$
- $\int \rho^\alpha(x,y) p_t(x,dy) \le C \cdot t^{1+\beta} \qquad (\forall x \in M, t \le 1)$
- (N,d) *satisfies a reverse variance inequality with exponent α.*

(i) Then for all $x \in M, t \in \mathbb{R}_+$ and $f \in \mathrm{Lip}(M,N)$

$$P_t^* f(x) = \lim_{n\to\infty} P_{t/n}^n f(x)$$

exists. The convergence is uniform in x, locally uniform in t and locally uniform in f (more precisely, uniform on $\{f \in \mathrm{Lip}(M,N) : \mathrm{dil} f \le n\}$ w.r.t. d_∞ for each n).

(ii) The limit is continuous in x, t and f. More precisely,

- $\mathrm{dil} P_t^* f \le e^{C(t+1)} \cdot \mathrm{dil} f$
- $d_\infty(P_t^* f, P_t^* g) \le d_\infty(f,g)$
- $d_\infty(P_s^* f, P_t^* f) \le C^{\frac{1}{\alpha}} \cdot \mathrm{dil} f \cdot |t-s|^{\frac{1+\beta}{\alpha}}$

(iii) $(P_t^*)_{t\in\mathbb{R}_+}$ is a strongly continuous semigroup on $\mathrm{Lip}(M,N)$ (equipped with a uniform pseudo metric d_∞) and

$$P_t^* f(x) = \lim_{s\to 0} P_s^{\lfloor t/s \rfloor} f(x) \tag{43}$$

uniformly in x, uniformly in t and locally uniformly in f.

Proof. (i) The assumptions on (p_t) imply that for each $f \in \mathrm{Lip}(M,N), t > 0$ and $x \in M$

$$\begin{aligned} v_{t,n}^{(\alpha)} f(x) &\le C' \cdot e^{C'\cdot t} (\mathrm{dil} f)^\alpha \cdot \sum_{i=0}^{n-1} \int d^\alpha(y,z) p_{\frac{t}{n}}(y,dz) p_{\frac{i}{n}t}(x,dy) \\ &\le C'' \cdot e^{C'\cdot t} (\mathrm{dil} f)^\alpha \cdot \frac{t^{1+\beta}}{n^\beta} \end{aligned}$$

According to the proof of Proposition 8.9 this implies

$$\begin{aligned} & d^2\left(P_{t/n}^n f, P_{1/m}^m f\right)(x) \\ \le\ & 2d^2\left(P_{t/n}^n f, P_{t/(2^k n)}^{2^k n} f\right)(x) + 2d^2\left(P_{1/m}^m f, P_{t/(2^k m)}^{2^k m} f\right)(x) \\ \le\ & 2c \cdot \left[\sum_{i=1}^\infty v_{t,2^i n}^{(\alpha)} f(x) + \sum_{i=1}^\infty v_{t,2^i m}^{(\alpha)} f(x)\right] \\ \le\ & C''' \cdot e^{C'\cdot t} \cdot (\mathrm{dil} f)^\alpha \cdot t^{1+\beta} \cdot \left(\frac{1}{n^\beta} + \frac{1}{m^\beta}\right) \end{aligned}$$

which proves the (locally) uniform convergence.

(ii) The continuity results in x and f are obvious (cf. Theorem 4.3 and Lemma 3.2(i)). Making use of the semigroup property from part (iii), it suffices to prove the continuity in t for $s = 0$.
But according to Lemma 3.2(ii)

$$\begin{aligned} d_\infty(f, P_t^* f) &\leq \sup_x \int d(f(x), f(y)) p_t(x, dy) \\ &\leq \operatorname{dil} f \cdot \sup_x \int \rho(x,y) p_t(x, dy) \leq \operatorname{dil} f \cdot \left(C \cdot t^{1+\beta}\right)^{\frac{1}{\alpha}} \end{aligned}$$

(iii) Part(i) already implies the semigroup property of $(P_t^*)_{t\in\mathbb{Q}_+}$. Indeed, for $s, t \in \mathbb{Q}_+$ choose $i, j, k \in \mathbb{N}$ with $s = \frac{i}{k}, t = \frac{j}{k}$. Then $P_t^* = \lim_{n\to\infty} P_{1/(kn)}^{jn}$ and $P_s^* = \lim_{m\to\infty} P_{1/(km)}^{im}$. Hence,

$$P_t^*(P_s^* f) = \lim_{n\to\infty} P_{1/(kn)}^{jn} \left(\lim_{m\to\infty} P_{1/(km)}^{im} f\right) = \lim_{n\to\infty} P_{1/(kn)}^{(j+i)n} = P_{s+t}^* f$$

Part (ii) implies that $(P_t^*)_{t\in\mathbb{Q}_+}$ is strongly continuous and thus the semigroup property (and the continuity) extends from $\mathbb{Q}_+$ to $\mathbb{R}_+$.
Finally, we are going to prove (43). Fix $t > 0$ and put $n_s := \lfloor \frac{t}{s} \rfloor, t_s := s \cdot n_s$ for $s > 0$. Note that $n_s \to \infty$ and $t_s \to t$ for $s \to 0$. Hence,

$$d\left(P_t^* f, P_s^{t/s} f\right) \leq d\left(P_t^* f, P_{t/n_s}^{n_s} f\right) \leq d\left(P_{t/n_s}^{n_s} f, P_{t_s/n_s}^{n_s} f\right) \to 0$$

for $s \to 0$. □

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