Metric Spaces of Lower Bounded Curvature

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Abstract. We study lower curvature bounds for metric spaces. For instance, an arbitrary metric space (N,d) is said to have curvature $\geq -\mathrm{K}$ (with $\mathrm{K} > 0$) iff

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \cosh\left(\sqrt{\mathrm{K}} \cdot d(y_i, y_j)\right) \leq \left(\sum_{i=1}^{k} \lambda_i \cosh\left(\sqrt{\mathrm{K}} \cdot d(y_0, y_i)\right)\right)^2$$

for all $k \in \mathbb{N}$, all $y_0, y_1, \ldots, y_k \in N$ and all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$. Similarly, the space (N,d) is said to have curvature ≥ 0 iff

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \, d^2(y_i, y_j) \leq 2 \sum_{i,j=1}^{k} \lambda_i \lambda_j \, d^2(y_0, y_i)$$

for k, y_i and λ_i as before. Using recent results of U. Lang and V. Schroeder, we prove that for complete geodesic spaces these lower curvature bounds coincide with lower curvature bounds in the sense of A. D. Alexandrov (based on triangle comparison) and with lower curvature bounds in the sense of Burago, Gromov, Perelman (based on quadruple comparison). Note, however, that in our approach (N,d) is neither assumed to be complete nor to be an inner metric space.

Our main result states that any metric space (N,d) of curvature $\geq -\mathrm{K}$ can be transformed into a metric space (N, d_{K}) of curvature ≥ 0 by replacing the metric $d(x,y)$ by the metric

$$d_{\mathrm{K}}(x,y) = \left(\frac{2}{\mathrm{K}} \log\cosh\left(\sqrt{\mathrm{K}} \cdot d(x,y)\right)\right)^{1/2}.$$

We also prove a partial converse, valid for Riemannian manifolds (N,d). Namely, if the (transformed) space (N, d_{K}) has curvature ≥ 0 then the original space must have curvature $\geq -4\mathrm{K}$.

1 Spaces of Curvature $\geq$ K

Let (N,d) be a metric space. For any curve (= continuous map) $\sigma : [a,b] \to N$ we define the *length* by

$$L(\sigma) = \sup \sum_{i=1}^{k} d\left(\sigma(t_{i-1}), \sigma(t_i)\right)$$

(being a number in $[0,\infty]$) where the supremum is taken over all $k \in \mathbb{N}$ and all subdivisions $a = t_0 < t_1 < \ldots < t_k = b$. Obviously, one always has $L(\sigma) \geq d(\sigma(a), \sigma(b))$. A curve $\sigma : [a,b] \to N$ is called *minimizing* if

$$L(\sigma) = d(\sigma(a), \sigma(b)).$$

The metric space (N,d) is called *(locally) geodesic space* if (every point $z \in X$ has a neighborhood U such that) for all $x, y \in X$ (or all $x, y \in U$, respectively) there exists a minimizing curve $\sigma : [0,1] \to N$ with $\sigma(0) = x, \sigma(1) = y$. In this case, we may assume without restriction that σ is a *minimizing geodesic* which means that it is minimizing and has constant speed in the sense that

$$d(\sigma(s), \sigma(t)) = c \cdot |s - t|$$

for all $s, t \in [a,b]$ and some constant $c \in \mathbb{R}$. The metric space (N,d) is called *inner metric space* or *length space* if

$$d(x,y) = \inf \left\{ L(\sigma) : \ \sigma \in C([0,1]\, N), \sigma(0) = x, \sigma(1) = y \right\}$$

for all $x, y \in N$.

In the sequel we will compare the metric sapce (N,d) with the 2-dimensional simply connected complete model spaces N_{K} of constant sectional curvature $\mathrm{K} \in \mathbb{R}$. Before doing this let us introduce some notations. For $\mathrm{K} > 0$ we put $D_{\mathrm{K}} = \frac{\pi}{\sqrt{\mathrm{K}}}$ and for $\mathrm{K} \leq 0$ we put $D_{\mathrm{K}} = \infty$. Then $\mathrm{diam}(N_{\mathrm{K}}) = D_{\mathrm{K}}$. Moreover, for $\mathrm{K} \in \mathbb{R}$ and $r \geq 0$ we put

$$\mathsf{S}_{\mathrm{K}} r = \begin{cases} \frac{1}{\sqrt{\mathrm{K}}} \cdot \sin(\sqrt{\mathrm{K}} r), & \text{if } \mathrm{K} > 0 \\ r, & \text{if } \mathrm{K} = 0 \\ \frac{1}{\sqrt{-\mathrm{K}}} \cdot \sinh(\sqrt{-\mathrm{K}} r), & \text{if } \mathrm{K} < 0 \end{cases}$$

and $\mathsf{C}_{\mathrm{K}} r = \mathsf{S}'_{\mathrm{K}} r$. Finally, for any three points $x, y, z \in N$ satisfying $d(x,y) + d(y,z) + d(z,x) < 2D_{\mathrm{K}}$ and $d(z,x) \cdot d(z,y) > 0$ we put

$$\mathsf{R}_0(z; x, y) = \frac{d^2(z,x) + d^2(z,y) - d^2(x,y)}{2 \cdot d(z,x) \cdot d(z,y)}$$

and, if $\mathrm{K} \neq 0$,

$$\mathsf{R}_{\mathrm{K}}(z; x, y) = \frac{\mathsf{C}_{\mathrm{K}} d(x,y) - \mathsf{C}_{\mathrm{K}} d(z,x) \cdot \mathsf{C}_{\mathrm{K}} d(z,y)}{\mathrm{K} \cdot \mathsf{S}_{\mathrm{K}} d(z,x) \cdot \mathsf{S}_{\mathrm{K}} d(z,y)}.$$

Note that for any such points $x, y, z \in N$ there exists an isometry $f : \{x, y, z\} \to N_{\mathrm{K}}$ and one can define the *comparison angle* $\angle_{\mathrm{K}}(z; x, y) \in [0, \pi]$ to be the angle subtended by the two geodesic segments in N_K connecting $f(z)$ with $f(x)$ and $f(y)$, respectively. It is well-known that

$$\mathsf{R}_K(z;x,y) = \cos \angle_K(z;x,y)$$

which implies that $\mathsf{R}_K(z;x,y) \in [-1,+1]$. In order to have everything well-defined, we put $\mathsf{R}_K(z;x,y) = +\infty$ whenever $d(x,y)+d(y,z)+d(z,x) \geq 2D_K$ or $d(z,x)\cdot d(z,y) = 0$.

1.1 DEFINITION. We say that a metric space (N,d) has *locally curvature* $\geq$ K if any point $x \in N$ has a neighborhood U such that for all $k \in \mathbb{N}$, for all $(z,y_1,\dots,y_k) \in U^{k+1}$ and for all $(\lambda_1,\dots,\lambda_k) \in [0,\infty[^k$

$$\sum_{i,j=1}^{k} \lambda_i\lambda_j \mathsf{R}_K(z;y_i,y_j) \geq 0. \tag{1.1}$$

We say that (N,d) has *curvature* $\geq$ K if this is true for $U=N$.

Note that (N,d) is neither assumed to be complete nor to be an inner metric space. Actually, in order to define the notion "curvature $\geq$ K" it suffices that (N,d) is a *pseudo metric space.*

Of course, it suffices to verify (1.1) for all points $z,y_1,\dots,y_k \in U$ with $d(z,y_i)+d(z,y_j)+d(y_i,y)j) < 2D_K$ and $d(z,y_i)\cdot d(z,y_j) > 0$.

The denominator of R_K appeats (up to its sign) only for conventional reasons. That is, in the case K $\neq 0$, the function $\mathsf{R}_K(z;x,y)$ in (1.1) can equivalently be replaced by the function $\mathsf{R}^*_K(z;x,y) = \frac{1}{K}[\mathsf{C}_K d(x,y) - \mathsf{C}_K d(z,x)\cdot \mathsf{C}_K d(z,y)]$ which amounts to

$$\frac{1}{K}\sum_{i,j=1}^{k} \lambda_i\lambda_j \mathsf{C}_K d(y_i,y_j) \geq \frac{1}{K}\left(\sum_{i=1}^{k}\lambda_i \mathsf{C}_K d(z,y_i)\right)^2 \tag{1.2}$$

for all $k \in \mathbb{N}$, all $(z,y_1,\dots,y_k) \in U^{k+1}$ and all $(\lambda_1,\dots,\lambda_k) \in [0,\infty[^k$. Similarly, in the case K $=0$ the function $\mathsf{R}_K(z;x,y)$ can be replaced by the function $\mathsf{R}^*_0(z;x,y) = d^2(z,x)+d^2(z,y)-d^2(x,y)$ which amounts to

$$\sum_{i,j=1}^{k} \lambda_i\lambda_j\, d^2(y_i,y_j) \leq 2\sum_{i=1}^{k}\lambda_i\, d^2(z,y_i) \tag{1.3}$$

for all $k \in \mathbb{N}$, all $(z,y_1,\dots,y_k) \in U^{k+1}$ and all $(\lambda_1,\dots,\lambda_k) \in [0,\infty[^k$ satisfying $\sum_{i=1}^k \lambda_i = 1$.

We easily obtain the following simple, but important example.

1.2 EXAMPLE. *Let (N,d) be any metric space. Then the metric space $(N,\sqrt{d})$ has curvature ≥ 0.*

However, $(N,\sqrt{d})$ is not an inner metric space if it contains more than one point.

1.3 PROPOSITION.

(i) *If a metric space (N,d) has (locally) curvature $\geq$ K then so has any subspace $(N_0, d|_{N_0})$ (for any $N_0 \subset N$ and $d|_{N_0}$ being the restriction of d on N_0).*

(ii) *If (N, d) has curvature $\geq \mathrm{K}$ for some $\mathrm{K} \leq 0$ then it also has curvature $\geq \mathrm{K}'$ for any $\mathrm{K}' \leq \mathrm{K}$.*

If (N, d) has locally curvature $\geq \mathrm{K}$ for some $\mathrm{K} \in \mathbb{R}$ then it also has locally curvature $\geq \mathrm{K}'$ for any $\mathrm{K}' \leq \mathrm{K}$.

(iii) *If (N, d) has (locally) curvature $\geq \mathrm{K}$ then for any $\alpha > 0$ the rescaled space $(N, \alpha \cdot d)$ has (locally) curvature $\geq \alpha^2 \cdot \mathrm{K}$.*

(iv) *Let $f : M \to N$ be any map from a set M into a (pseudo) metric space (N, d) and define a pseudo metric d_* on M by $d_*(x, y) = d(f(x), f(y))$.*
If (N, d) has curvature $\geq \mathrm{K}$ then (M, d_*) has curvature $\geq \mathrm{K}$.

Proof. (i), (iv) Obvious.

(ii) This follows from the fact that $\mathrm{K}' \leq \mathrm{K}$ obviously implies $\angle_{\mathrm{K}'}(z; x, y) \leq \angle_{\mathrm{K}}(z; x, y)$ for all $x, y, z \in N$.

(iii) If one replaces d by $\lambda \cdot d$ then R_{K} becomes $\gamma^2 \cdot \mathsf{R}_{\gamma^2 \cdot \mathrm{K}}$. □

If (N, d) is a (locally) geodesic space then a triangle in N is a triple $\Delta = (\sigma^1, \sigma^2, \sigma^3)$ of minimizing geodesics $\sigma^i : [0, 1] \to N$ whose endpoint match as usual. If Δ has perimeter $P(\Delta) := \sum_i L(\sigma^i) < 2D_{\mathrm{K}}$ then there exists a *comparison triangle* $\Delta_{\mathrm{K}} = (\sigma^1_{\mathrm{K}}, \sigma^2_{\mathrm{K}}, \sigma^3_{\mathrm{K}})$ in N_{K} (uniquely determined up to isometry) such that $L(\sigma^i) = L(\sigma^i_{\mathrm{K}})$ for all $i = 1, 2, 3$.

1.4 THEOREM. *Let (N, d) be a locally geodesic space. Then for any $\mathrm{K} \in \mathbb{R}$ the following are equivalent:*

(i) *(N, d) has locally curvature $\geq \mathrm{K}$.*

(ii) *Every point $y \in N$ has a neighborhood $U \subset N$ such that all quadruples (z, x_1, x_2, x_3) of points in U satisfy the following condition:*

$$\angle_{\mathrm{K}}(z; x_1, x_2) + \angle_{\mathrm{K}}(z; x_2, x_3) + \angle_{\mathrm{K}}(z; x_3, x_1) \leq 2\pi$$

whenever all angles are defined.

(iii) *Every point $y \in N$ has a neighborhood $U \subset N$ such that all triangles $\Delta = (\sigma^1, \sigma^2, \sigma^3)$ with vertices in U and perimeter $< 2D_{\mathrm{K}}$ satisfy*

$$d(\sigma^i(s), \sigma^j(t)) \geq d(\sigma^i_{\mathrm{K}}(s), \sigma^j_{\mathrm{K}}(t))$$

for all $s, t \in [0, 1]$ and all $i, j \in \{1, 2, 3\}$.

Proof. The equivalence of (ii) and (iii) was proven by BURAGO, GROMOV, PERELMAN (1992). The implication (iii) $\Rightarrow$ (i) is due to LANG, SCHROEDER (1996). Let us repeat their argument. Let us fix a point $z \in N$. We make use of the tangent cone $(T_z N, d_z)$ of N at z (in the sense of LANG, SCHROEDER (1996)). Choose $k \in \mathbb{N}$ and k points $y_1, \ldots, y_k \in N$. Without restriction we may assume that these points are pairwise disjoint and that $d(z, y_i) + d(z, y_j + d(y_i, y_j) < 2D_{\mathrm{K}}$ for all $i, j = 1, \ldots, k$. For $i = 1, \ldots, k$ let $\sigma_i : [0, t_i] \to M$ be a unit speed geodesic connecting $z = \sigma_i(0)$ and $y_i = \sigma_i(t_i)$ and let

$v_i = \dot{\sigma}_i(0) \in T_zN$. Then $d(z, y_i) = t_i$ and $\|v_i\| = d_z(0, v_i) = 1$. According to LANG, SCHROEDER (1996), Prop. 3.2, the matrix $(\langle v_i, v_j \rangle)_{i,j}$ satisfies

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \langle v_i, v_j \rangle \geq 0 \tag{1.4}$$

for all $\lambda_1, \ldots \lambda_k \in [0, \infty[$. (Note that by definition this means that at the origin the space (T_zN, d_z) has curvature ≥ 0 in the sense of our Definition 1.1.) Now we use the assumption (iii). It implies that

$$\langle v_i, v_j \rangle \leq \mathsf{R}_{\mathsf{K}}(z; y_i, y_j).$$

Together with (1.4) this yields the assertion (i)

Now let us prove the implication (i) $\Rightarrow$ (ii). Let points $z, x_1, x_2, x_3 \in N$ be given with comparison angles $\gamma_i = \angle_{\mathrm{K}}(z; x_{i+1}, x_{i+2})$ (indices mod 3). Without restriction assume $0 < \gamma_i < \pi$ for all $i = 1, 2, 3$. Now let us assume (i), i.e.

$$\sum_{i,j=1}^{3} \lambda_i \lambda_j \cos \angle_{\mathrm{K}}(z; x_i, x_j) \geq 0$$

or, equivalently,

$$\sum_{i}^{3} \lambda_i^2 + 2 \sum_{i=1}^{3} \lambda_i \lambda_{i+1} \cos \gamma_{i+2} \geq 0 \tag{1.5}$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in [0, \infty[^3$ and assume that (ii) is not satisfied, i.e.

$$\sum_{i=1}^{3} \gamma_i > 2\pi. \tag{1.6}$$

Let $\epsilon = \sum_{i=1}^{3} \gamma_i - 2\pi$ and put $\gamma_1^* = \gamma_1, \gamma_2^* = \gamma_2$ and $\gamma_3^* = \gamma_3 - \epsilon$. Then $\sum_{i=1}^{3} \gamma_i^* = 2\pi$. Hence, Euclidean geometry implies that

$$\sum_{i}^{3} \lambda_i^2 + 2 \sum_{i=1}^{3} \lambda_i \lambda_{i+1} \cos \gamma_{i+2}^* \geq 0 \tag{1.7}$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in [0, \infty[^3$ with equality in (1.7) for a one-parameter family of $(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in]0, \infty[^3$. (In order to see this, choose three points $x_1^*, x_2^*, x_3^* \in \mathbb{R}^2$ with $|x_i^*| = 1$ and $\gamma_i^* = \angle_0(0; x_{i+1}^*, x_{i+2}^*)$ (indices mod 3). Then the LHS of (1.7) is just $\left(\sum_{i=1}^{3} \lambda_i x_i\right)^2$ which of course is nonnegative and vanishes iff $\sum_{i=1}^{3} \lambda_i x_i = 0$.) For these λ_i^* we get

$$\sum_{i}^{3} (\lambda_i^*)^2 + 2\lambda_i^* \lambda_{i+1}^* \cos \gamma_{i+2} < \sum_{i}^{3} (\lambda_i^*)^2 + 2\lambda_i^* \lambda_{i+1}^* \cos \gamma_{i+2}^* = 0$$

which is a contradiction to (1.5). $\square$

1.5 COROLLARY. *Let (N, d) be a complete geodesic space. Then for any $\mathrm{K} \in \mathbb{R}$ the following are equivalent:*

(i) (N,d) *has locally curvature* $\geq \mathrm{K}$.

(ii) (N,d) *has curvature* $\geq \mathrm{K}$.

(iii) *All quadruples* (z, x_1, x_2, x_3) *of points in* N *satisfy the following condition:*

$$\angle_{\mathrm{K}}(z; x_1, x_2) + \angle_{\mathrm{K}}(z; x_2, x_3) + \angle_{\mathrm{K}}(z; x_3, x_1) \leq 2\pi$$

whenever all angles are defined.

(iv) *All triangles* $\Delta = (\sigma^1, \sigma^2, \sigma^3)$ *with vertices in* N *and perimeter* $< 2D_{\mathrm{K}}$ *satisfy*

$$d(\sigma^i(s), \sigma^j(t)) \geq d(\sigma^i_{\mathrm{K}}(s), \sigma^j_{\mathrm{K}}(t))$$

for all $s,t \in [0,1]$ *and all* $i,j \in \{1,2,3\}$.

Proof. According to BURAGO, GROMOV, PERELMAN (1992), conditions (iii) and (iv) are equivalent to conditions (ii) and (iii) of the previous Theorem 1.4 which have been proved to be equivalent to (i). The same proof yields now that (iii)+(iv) imply (ii). The implication (ii) $\Rightarrow$ (i) is trivial. □

1.6 COROLLARY. *Let* (N,h) *be a Riemannian manifold and let* d *denote the associated Riemannian metric. Then for any* $\mathrm{K} \in \mathbb{R}$ *the following are equivalent:*

(i) *The Riemannian manifold* (N,h) *has sectional curvature* $\geq \mathrm{K}$.

(ii) *The metric space* (N,d) *has locally curvature* $\geq \mathrm{K}$ *in the sense of Definition 1.1.*

If in addition N *is complete then (i) and (ii) are equivalent to*

(iii) *The metric space* (N,d) *has curvature* $\geq \mathrm{K}$ *in the sense of Definition 1.1.*

1.7 PROPOSITION. *Assume that the metric space* (N,d) *is Polish (i.e. separable and complete metrizable). Then for any* $\mathrm{K} \in \mathbb{R}$ *the following are equivalent:*

(i) (N,d) *has curvature* $\geq \mathrm{K}$.

(ii) *For all* $z \in N$ *and all finite measures* ν *on* N *(equipped with its Borel* σ*-field)*

$$\int_N \int_N \mathsf{R}_{\mathrm{K}}(z; x, y))\, \nu(dx)\nu(dy) \geq 0 \tag{1.8}$$

(iii) *For all* $z \in N$ *and all probability measures* ν *on* N

$$\frac{1}{\mathrm{K}} \int_N \int_N \mathsf{C}_{\mathrm{K}} d(x,y) \nu(dy)\, \nu(dx) \geq \frac{1}{\mathrm{K}} \left(\int_N \mathsf{C}_{\mathrm{K}} d(z,x)\, \nu(dx) \right)^2 \tag{1.9}$$

if $\mathrm{K} \neq 0$ *and if* $\mathrm{K} = 0$

$$\int_N \int_N d^2(x,y) \nu(dy)\, \nu(dx) \leq 2 \int_N d^2(z,x)\, \nu(dx). \tag{1.10}$$

Proof. Let us fix $z \in N$ and put $Z = \{(x,y) \in N \times N : d(z,x) + d(z,y) + d(x,y) < 2D_{\mathrm{K}} \text{ and } d(z,x) \cdot d(z,y) > 0\}$. Without restriction, we may assume that ν is probability measure on N with $(\nu \otimes \nu)(N \times N \setminus Z) = 0$ (otherwise the LHS in (1.8) is $+\infty$ and the claim is trivial). Since the space $(Z, d \times d)$ is Polish, there exists, for any $n \in \mathbb{N}$, a compact set $Z_n \subset Z$ such that $(\nu \otimes \nu)(Z \setminus Z_n) \leq \frac{1}{n}$. Hence,

$$\int_{Z_n} \mathsf{R}_{\mathrm{K}}(z;x,y)\nu(dx)\nu(dy) \stackrel{n\to\infty}{\longrightarrow} \int_{Z} \mathsf{R}_{\mathrm{K}}(z;x,y)\nu(dx)\nu(dy) = \int_{N\times N} \mathsf{R}_{\mathrm{K}}(z;x,y)\nu(dx)\nu(dy) < \infty.$$

Since the set Z_n is compact and since the function $\mathsf{R}_{\mathrm{K}}(z;.)$ is continuous, for any $\epsilon > 0$ there exist $k \in \mathbb{N}$, $(\lambda_1, \ldots, \lambda_k) \in [0,\infty[^k$ and $(y_1, \ldots, y_k) \in Z_n^k$ such that

$$\int_{Z_n} \mathsf{R}_{\mathrm{K}}(z;x,y)\nu(dx)\nu(dy) \geq \sum_{i,j=1}^{k} \lambda_i\lambda_j \mathsf{R}_{\mathrm{K}}(z;y_i,y_j) - \epsilon.$$

Together with (1.1) this yields the claim.

In order to prove (ii) $\Rightarrow$ (iii), let us (for simplicity) assume that $\mathrm{K} < 0$. (The other cases are treated in essentially the same way.) Let a point $z \in N$ and a probability measure ν be given. Without restriction, we may assume $\int_N \mathsf{C}_{\mathrm{K}} d(z,x)\nu(dx) < \infty$. (Otherwise, nothing is to prove.) Then define another finite measure $\tilde{\nu}$ on N by $\tilde{\nu}(dx) = \mathsf{S}_{\mathrm{K}} d(z,x)\,\nu(dx)$ and apply (ii) to this measure which yields

$$\begin{aligned} 0 &\leq \int_N \int_N \mathsf{R}_{\mathrm{K}}(z;x,y)\,\tilde{\nu}(dy)\,\tilde{\nu}(dx) \\ &= \frac{1}{\mathrm{K}} \int_N \int_N [\mathsf{C}_{\mathrm{K}} d(x,y) - \mathsf{C}_{\mathrm{K}} d(z,x) \cdot \mathsf{C}_{\mathrm{K}} d(z,y)]\,\nu(dy)\,\nu(dx) \\ &= \frac{1}{\mathrm{K}} \left[\int_N \int_N \mathsf{C}_{\mathrm{K}} d(x,y)\nu(dy)\,\nu(dx) - \left(\int_N \mathsf{C}_{\mathrm{K}} d(z,x)\,\nu(dx) \right)^2 \right]. \end{aligned}$$

□

1.8 Proposition. *Let $(N, \|.\|)$ be a real normed space and let d denote the associated metric on N. Then for any $\mathrm{K} \in \mathbb{R}$ the following are equivalent:*

(i) *(N,d) has locally curvature $\geq \mathrm{K}$;*

(ii) *$(N, \|.\|)$ is a prehilbert space (i.e. there exists an inner product $\langle .,. \rangle$ on N which induces the norm $\|.\|$) and $\mathrm{K} \leq 0$;*

(iii) *(N,d) has curvature ≥ 0 and ≤ 0 (in the sense of A.D.Alexandrov).*

Moreover, if (N,d) has locally curvature $\geq \mathrm{K}$ for some $\mathrm{K} > 0$ then $\dim(N) = 1$.

Proof. If (N,d) has (locally) curvature $\geq \mathrm{K}$ for some $\mathrm{K} \in \mathbb{R}$ then by scaling it follows that it has curvature ≥ 0. It has curvature ≥ 0 if and only if it is a prehilbert space (see for instance Sturm (1997)) and in this case it obviously also has curvature ≤ 0. Finally, if (N,d) has locally curvature $\geq \mathrm{K}$ for some $\mathrm{K} > 0$ then by scaling it follows that (N,d) has locally curvature $\geq K'$ for any K'. □

2 Transformation into Spaces of Curvature ≥ 0

Now let us restrict to spaces of curvature $\geq -\mathrm{K}$ for some $\mathrm{K} \geq 0$. For $\mathrm{K} > 0$ we put

$$\Phi_{\mathrm{K}}(r) = \frac{1}{K} \log\cosh\left(\sqrt{\mathrm{K}} \cdot r\right). \tag{2.1}$$

Note that for K fixed $\Phi_{\mathrm{K}}(r) \approx \frac{r^2}{2}$ for $r \to 0$ and $\Phi_{\mathrm{K}}(r) \approx \frac{r}{\sqrt{\mathrm{K}}}$ for $r \to \infty$. On the other hand, for $r \geq 0$ fixed, $\mathrm{K} \to \Phi_{\mathrm{K}}(r)$ is decreasing on $[0, \infty[$. Moreover, we define

$$d_{\mathrm{K}}(x, y) = \sqrt{2\Phi_{\mathrm{K}}(d(x, y))} \tag{2.2}$$

for $x, y \in N$. One easily verifies that d_{K} is a *metric* on N which satisfies

$$\sqrt{\frac{2}{\sqrt{\mathrm{K}}} d(x, y) - \frac{2}{\mathrm{K}} \log 2} \;\leq\; d_{\mathrm{K}}(x, y) \;\leq\; d(x, y)$$

for $K > 0$ and all $x, y \in N$. Moreover, $d_{\mathrm{K}}(x, y) \approx d(x, y)$ if $d(x, y) \ll 1$ and $d_{\mathrm{K}}(x, y) \approx (4/\mathrm{K})^{1/4} \cdot \sqrt{d(x, y)}$ if $d(x, y) \gg 1$.

If N consists of more than one point, then d_{K} is not an inner metric. Its associated inner metric coincides with the inner metric associated with d. In particular, if the original metric d is an inner metric then the inner metric associated with d_{K} is just d itself.

Finally, we put $\Phi_0(r) = r^2/2$ and $d_0 = d$.

Now let (N, d) be any metric space and let K be any nonnegative number.

2.1 Theorem. *If the metric space (N, d) has curvature $\geq -\mathrm{K}$ then the metric space (N, d_{K}) has curvature ≥ 0.*

Proof. In the case $\mathrm{K} = 0$, nothing is to prove. Assume that $\mathrm{K} > 0$. Then we have to prove that

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \log\left(\frac{\mathsf{C}_{-\mathrm{K}} d(y_i, y_j)}{\mathsf{C}_{-\mathrm{K}} d(z, y_i) \cdot \mathsf{C}_{-\mathrm{K}} d(z, y_j)}\right) \leq 0$$

for all $(\lambda_1, \ldots, \lambda_k) \in [0, \infty[^k$ with $\sum_{i=1}^k \lambda_i = 1$. But Jensen's inequality together with property (2.1) imply

$$\begin{aligned}
&\sum_{i,j=1}^{k} \lambda_i \lambda_j \log\left(\frac{\mathsf{C}_{-\mathrm{K}} d(y_i, y_j)}{\mathsf{C}_{-\mathrm{K}} d(z, y_i) \cdot \mathsf{C}_{-\mathrm{K}} d(z, y_j)}\right) \\
&\leq \log\left(\sum_{i,j=1}^{k} \lambda_i \lambda_j \frac{\mathsf{C}_{-\mathrm{K}} d(y_i, y_j)}{\mathsf{C}_{-\mathrm{K}} d(z, y_i) \cdot \mathsf{C}_{-\mathrm{K}} d(z, y_j)}\right) \\
&= \log\left(1 - \mathrm{K} \cdot \sum_{i,j=1}^{k} \frac{\lambda_i \cdot \mathsf{S}_{-\mathrm{K}} d(z, y_i)}{\mathsf{C}_{-\mathrm{K}} d(z, y_i)} \cdot \frac{\lambda_j \cdot \mathsf{S}_{-\mathrm{K}} d(z, y_j)}{\mathsf{C}_{-\mathrm{K}} d(z, y_j)} \cdot \mathsf{R}_{-\mathrm{K}}(z; y_i, y_j)\right) \\
&\leq 0.
\end{aligned}$$

□

2.2 COROLLARY. *Assume that the metric space (N,d) is Polish and has curvature $\geq -\mathrm{K}$. Then for all $z \in N$ and all probability measures ν on N*

$$\int_N \Phi_{\mathrm{K}}(d(z,x))\,\nu(dx) \geq \frac{1}{2}\int_N\int_N \Phi_{\mathrm{K}}(d(x,y))\,\nu(dx)\nu(dy). \tag{2.3}$$

Proof. Replacing $\Phi_{\mathrm{K}} d$ by $d_{\mathrm{K}}^2/2$ the claim follows immediately from Theorem 2.1 and Proposition 1.7. □

This result has important consequences for the approximation of energy functionals for mappings into the metric space (N,d) ("target space"). In order to illustrate this, let a measure space $(M,\mathcal{M},\mu)$ ("domain space") be given and on this space a symmetric Markovian semigroup $(P_t)_{t>0}$. See STURM (1997) for details. Moreover, we fix once for all an arbitrary number $t_0 \in]0,\infty[$ and put $t_n = 2^{-n}t_0$ for $n \in \mathbb{N}$.

A map $f: M \to N$ will be called measurable if it is measurable with respect to the given σ-field $\mathcal{M}$ on M and the Borel σ-field $\mathcal{N}$ on N. For each such map $f: M \to N$, each $\mathrm{K} \geq 0$ and each $n \in \mathbb{N}$ we define the *approximated energy*

$$E_{\mathrm{K}}^n(f) = \frac{1}{t_n}\int_M\int_M \Phi_{\mathrm{K}}(d(f(x),f(y)))P_{t_n}(x,dy)\mu(dx). \tag{2.4}$$

A detailed analysis of these functionals will be carried out in STURM (1997A). Among others, we show that under suitable assumptions

$$\lim_{n\to\infty} E_{\mathrm{K}}^n(f) = \lim_{t\to 0}\frac{1}{2t}\int_M\int_M d^2(f(x),f(y))P_t(x,dy)\mu(dx).$$

Here we only state the important monotonicity property of these approximations.

2.3 COROLLARY. *Assume that the metric space (N,d) is Polish and has curvature $\geq -\mathrm{K}$. Then for any measurable map $f: M \to N$ the sequence $E_{\mathrm{K}}^n(f)$ is increasing in $n \in \mathbb{N}$.*

Proof. By definition of Φ_{K} and d_{K}

$$E_{\mathrm{K}}^n(f) = \frac{1}{2t_n}\int_M\int_M d_{\mathrm{K}}^2(f(x),f(y))P_{t_n}(x,dy)\mu(dx). \tag{2.5}$$

According to the main result of STURM (1997), the RHS of (2.5) is increasing in n provided (N,d_{K}) has curvature ≥ 0. This in turn follows from Theorem 2.1. □

Now let us ask for the converse to Theorem 2.1.

2.4 THEOREM. *Let (N,h) be a Riemannian manifold of dimension $n \geq 2$, let d denote the associated Riemannian metric and let K be any nonnegative number.*

If the metric space (N,d_{K}) has curvature ≥ 0 then the the manifold N has curvature $\geq -4\mathrm{K}$.

Proof. Fix a point $z \in N$, a two-dimensional plane E in the tangent space T_zN and a number $K' < -R(z,E)$ where $R(z,E)$ denotes the sectional curvature at z in the plane

E. If $R(z,E) \geq 0$ then nothing is to prove. Hence, we may assume without restriction that $R(z,E) < 0$ and $K' > 0$. Given this K' we can choose a neighborhood U of z in N such that

$$K' < -R(x,E)$$

for all $x \in U$. Now let e_1, e_2 be a ONB of E and define for $r > 0$

$$y_1(r) = \exp_z(re_1), \quad y_2(r) = \exp_z(-\frac{1}{2}re_1 + \frac{1}{2}\sqrt{3}re_2), \quad y_3(r) = \exp_z(-\frac{1}{2}re_1 + \frac{1}{2}\sqrt{3}re_2).$$

Then by classical comparison theorems of Riemannian geometry

$$\cosh\left(\sqrt{\mathrm{K}'}\cdot d(y_i(r), y_j(r))\right) \geq \frac{3}{2}\cosh^2\left(\sqrt{\mathrm{K}'}r\right) - \frac{1}{2} \tag{2.5}$$

for all $i,j = 1,2,3$, $i \neq j$ and all sufficiently small $r > 0$.

On the other hand, the assertion that the metric space (N, d_{K}) has curvature ≥ 0 implies that

$$\frac{1}{9}\sum_{i\neq j}\log\cosh\left(\sqrt{\mathrm{K}}\cdot d(y_i(r), y_j(r))\right) \leq 2\log\cosh\left(\sqrt{\mathrm{K}}r\right)$$

for all $r > 0$. In particular,

$$\cosh\left(\sqrt{\mathrm{K}}\cdot d(y_i(r), y_j(r))\right) \leq \cosh^3\left(\sqrt{\mathrm{K}}r\right) \tag{2.6}$$

for some $i,j = 1,2,3$, $i \neq j$ and all $r > 0$. Estimates (2.5) and (2.6) together imply

$$f(r) \geq 0 \tag{2.7}$$

for all sufficiently small $r > 0$ where

$$f(r) := \frac{1}{\sqrt{\mathrm{K}}}\mathrm{arcosh}\left(\cosh^3\left(\sqrt{\mathrm{K}}r\right)\right) - \frac{1}{\sqrt{\mathrm{K}'}}\mathrm{arcosh}\left(\frac{3}{2}\cosh^2\left(\sqrt{\mathrm{K}'}r\right) - \frac{1}{2}\right).$$

Straightforward but lengthy calcluations yield

$$f(0) = f'(0) = f''(0) = 0 \qquad \text{and} \qquad f'''(0) = \sqrt{3}\,(K - K'/4). \tag{2.8}$$

Hence, (2.7) implies $K' \leq 4K$. Since K' can be chosen arbitrarily close to $-R(z,E)$ this implies

$$R(z,E) \geq -4\mathrm{K}.$$

Finally, since the point $z \in N$ and the plane $E \subset T_zN$ are arbitrary this implies the claim. □

2.5 COROLLARY. *Let (N,h) be a Riemannian manifold of dimension $n \geq 2$ and let d denote the associated Riemannian metric. Then the curvature of the manifold N is bounded from below if and only if for some $\mathrm{K} \geq 0$ the metric space (N, d_{K}) has curvature ≥ 0.*

Finally, we want to look for consequences of the "nonnegative curvature" of the metric space (N, d_K). In Riemannian geometry, one of the important consequences of nonnegative curvature is that

$$\frac{1}{2}\Delta d^2(.,y) \le n$$

(weakly on N and strongly) on $N \setminus \mathrm{Cut}(y)$ for each $y \in N$ where n denotes the dimension of the manifold N and $d^2(.,y)$ the function $x \mapsto d^2(x,y)$. (The Laplacian operates on the x-variable.) On the other hand, nonpositive curvature implies

$$\frac{1}{2}\Delta d^2(.,y) \ge n$$

(strongly) on $N \setminus \mathrm{Cut}(y)$ for each $y \in N$. In probabilistic language, this can be reformulated as follows: if $(X_t)_{0\le t<\zeta}$ denotes Brownian motion on N with (maximal) lifetime ζ and if $\zeta(y) = \{t \ge 0 : X_t \notin N \setminus \mathrm{Cut}(y)\}$ denotes its lifetime in $N \setminus \mathrm{Cut}(y)$ then

$$\left(d^2(X_t, y) - n \cdot t\right)_{0\le t<\zeta(y)}$$

is a local supermartingale (or submartingale) provided N has curvature ≥ 0 (or ≤ 0, resp.).

2.6 Proposition. *Let (N, h) be a Riemannian manifold of dimension n and let d denote the associated Riemannian metric.*

a) If the Ricci curvature is bounded from below by $-(n-1)\mathrm{K}$ (in particular, if the sectional curvature is bounded from below by $-\mathrm{K}$) for some $\mathrm{K} \ge 0$ then for every $y \in N$

$$\frac{1}{2}\Delta d_{\mathrm{K}}^2(.,y) \le n \tag{2.9}$$

weakly on N and strongly on $N \setminus \mathrm{Cut}(y)$ and for every Brownian motion $(X_t)_{0\le t<\zeta}$ on N the stochastic process

$$\left(d_{\mathrm{K}}^2(X_t, y) - n \cdot t\right)_{0\le t<\zeta} \tag{2.10}$$

is a local supermartingale.

b) If the sectional curvature is bounded from above by $-\mathrm{K}$ for some $\mathrm{K} \ge 0$ then for every $y \in N$

$$\frac{1}{2}\Delta d_{\mathrm{K}}^2(.,y) \ge n-1 \tag{2.11}$$

strongly on $N \setminus \mathrm{Cut}(y)$ and for every Brownian motion $(X_t)_{0\le t<\zeta}$ on N the stochastic process

$$\left(d_{\mathrm{K}}^2(X_t, y) - (n-1) \cdot t\right)_{0\le t<\zeta(y)} \tag{2.12}$$

(with $\zeta(y)$ being the lifetime within $N \setminus \mathrm{Cut}(y)$) is a local submartingale.

Proof. a) The claim follows immediately from the fact that for all $x, y \in N$ (and with $r = d(x,y)$)

$$\frac{1}{2}\Delta d_{\mathrm{K}}^2(x,y) \le \Phi_{\mathrm{K}}''(r) + (n-1)\sqrt{\mathrm{K}}\coth(\sqrt{\mathrm{K}}r)\cdot\Phi_{\mathrm{K}}'(r) = \frac{1}{\cosh^2(\sqrt{\mathrm{K}}r)} + (n-1) \le n.$$

b) Similarly, now the claim follows from

$$\frac{1}{2}\Delta d_K^2(x,y) \geq \Phi_K''(r) + (n-1)\sqrt{K}\coth(\sqrt{K}r)\cdot\Phi_K'(r) = \frac{1}{\cosh^2(\sqrt{K}r)} + (n-1) \geq n-1$$

for all $x \neq \mathrm{Cut}(y)$ (and with $r = d(x,y)$). □

It might be helpful to compare the function $\Phi_K(r)$ defined in (2.1) with the function

$$\Phi_{K,n}(r) = n\cdot\int_0^r\int_0^t \left[\frac{\sinh(\sqrt{K}s)}{\sinh(\sqrt{K}t)}\right]^{n-1} ds\, dt$$

(for $K > 0$ and $n \in \mathbb{N}$) which is the (unique) solution of the following Sturm-Liouville equation on $[0,\infty[$:

$$\Phi''(r) + (n-1)\sqrt{K}\cdot\coth(\sqrt{K}r)\cdot\Phi'(r) = n, \qquad \Phi'(0) = \Phi(0) = 0.$$

For instance, one obtains

$$\Phi_{K,1} = r^2/2, \quad \Phi_{K,2} = \frac{4}{K}\log\cosh(\sqrt{K}r/2), \quad \Phi_{K,3} = \frac{3}{2K}\left[\coth(\sqrt{K}r)\cdot\sqrt{K}r - 1\right].$$

One easily verifies

$$\Phi_K(r) \;\leq\; \Phi_{K,n}(r) \;\leq\; \frac{n}{n-1}\cdot\Phi_K(r).$$

Similarly as in (2.2) we define

$$d_{K,n}(x,y) = \sqrt{2\Phi_{K,n}(d(x,y))}.$$

2.7 PROPOSITION. *If in the situation of Proposition 2.5 the manifold N has constant sectional curvature then the following are equivalent:*

- *the curvature is $-K$;*
- *$\frac{1}{2}\Delta d_{K,n}^2(.,y) = n$ on $N \setminus \mathrm{Cut}(y)$ for every $y \in N$;*
- *$(d_{K,n}^2(X_t,y) - n\cdot t)_{0\leq t<\zeta(y)}$ is a local martingale for every $y \in N$ and every Brownian motion $(X_t)_{0\leq t<\zeta}$ on N.*

A remarkable consequence is that for manifolds of constant curvature

$$(N,d)\text{ curv. } \geq -K \;\Rightarrow\; (N,d_K)\text{ curv. } \geq 0 \;\Rightarrow\; (N,d)\text{ curv. } \geq -\left(\frac{n}{n-1}\right)^2\cdot K.$$

This should be compared with Theorems 2.1 and 2.4.

Another consequence of Proposition 2.7 is that by comparison the assertions of Proposition 2.6 are true with d_K replaced by $d_{K,n}$ where in (2.11) and (2.12) now the constant $n-1$ may be replaced by n.

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