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Is a Diffusion Process Determined by Its Intrinsic Metric?†

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Abstract—J. R. Norris proved that the small time asymptotic $\lim_{t\to 0} 2t\cdot \log p(t,x,y)$ of a symmetric elliptic diffusion on $\mathbb{R}^n$ (or, more general, on a Lipschitz manifold) is determined by the intrinsic metric defined in terms of the associated Dirichlet form. Here we ask the question: Is the Dirichlet form (or the diffusion process) determined uniquely by its intrinsic metric (i.e. by its small time asymptotic)?

The answer is NO. For any symmetric elliptic diffusion there exists another one with the same small time asymptotic but with strictly smaller diffusion coefficients.

However, the answer is YES if *a priori* we know that the diffusion coefficients are continuous.

1. INTRODUCTION

Throughout this paper, the state space X will be a fixed Lipschitz manifold of dimension $n \geq 2$ and m a Radon measure on it. In order to make the exposition more readable, we will concentrate on the case $X = \mathbb{R}^n$. However, we emphasize that all results presented in the sequel also hold true in the general case of X being a Lipschitz manifold (for the technical details we refer to Ref. [1]).

Now assume that $X = \mathbb{R}^n$, $n \geq 2$, $m(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ with a measurable function φ satisfying $\operatorname{ess\,inf}_K \varphi > 0$ and $\operatorname{ess\,sup}_K \varphi < \infty$ for each compact $K \subset X$. Let $\mathscr{A}$ denote the set of symmetric matrix-valued, measurable maps

$$a = (a_{ij})_{ij}: X \to \mathbb{R}^{n\times n}$$

which are elliptic in the sense that there exists a continuous function $\lambda:\mathbb{R} \to [1,\infty[$ such that

$$\frac{1}{\lambda(x)}\cdot|\xi|^2 \leq \langle \xi, a(x)\xi\rangle \leq \lambda(x)\cdot|\xi|^2$$

for a.e. $x \in X$ and every $\xi \in \mathbb{R}^n$ where here and henceforth

$$\langle \xi, a(x)\xi\rangle = \sum_{i,j=1}^{n} \xi_i a_{ij}(x)\xi_j.$$

2. THE INTRINSIC METRIC AND THE PATH METRIC

With any *diffusion matrix* $a \in \mathscr{A}$ we associate a *Dirichlet form* $(\varepsilon_a, \mathscr{D}(\varepsilon_a))$ with core $\mathscr{C}_0^{\mathrm{Lip}}(X)$ (the set of Lipschitz functions $f:X \to \mathbb{R}$ with compact support). For $f \in \mathscr{C}_0^{\mathrm{Lip}}(X)$ we put

$$\varepsilon_a(f) = \frac{1}{2}\int_X \langle \nabla f, a\nabla f\rangle(x) m(\mathrm{d}x).$$

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Then it is well-known that $(\varepsilon_a,\mathscr{C}_0^{\mathrm{Lip}}(X))$ is closable in $L^2(X,m)$ and its closure $(\varepsilon_a,\mathscr{D}(\varepsilon_a))$ is a regular, strongly local Dirichlet form. One easily verifies that $\mathscr{D}_{\mathrm{loc}}(\varepsilon_a)=W^{1,2}_{\mathrm{loc}}(X)$. See, for example, Fukushima–Oshima–Takeda [2], Norris [1] or Sturm [3].

The associated intrinsic metric d_a is defined by

$$d_a(x,y)=\sup\{f(x)-f(y):f\in\mathscr{D}_{\mathrm{loc}}(\varepsilon)\cap\mathscr{C}(X),\ \langle\nabla f,a\nabla f\rangle\leq 1 \text{ a.e. on } X\}. \tag{1}$$

It is the 'Riemannian metric' on X associated with $a\in\mathscr{A}$. Obviously, for each point $x\in X$ there exists a neighborhood on which the original (= Euclidean) metric and the intrinsic metric d_a are comparable. In particular, the topologies coincide. Given $a\in\mathscr{A}$ we define the *length* of a Lipschitz curve $\gamma:[0,1]\to X$ by

$$L_a(\gamma)=\left(\int_0^1\langle\dot\gamma(t),a^{-1}(\gamma(t))\dot\gamma(t)\rangle\mathrm{d}t\right)^{1/2}.$$

The *path metric* d_a^* is defined by

$$d_a^*(x,y)=\inf\{L_a(\gamma):\gamma:[0,1]\to X \text{ Lipschitz with } \gamma(0)=x,\ \gamma(1)=y\}.$$

One easily checks that $d_a^*(x,y)\leq d_a(x,y)$ for all x, $y\in X$, cf. Sturm [4] and Norris [5]. In general, however, there will be *no equality*! For instance, this follows from the fact that d_a only depends on the m-equivalence class of a (i.e. $d_a=d_{\tilde a}$ whenever $a(x)=\tilde a(x)$ for a.e. $x\in X$) whereas d_a^* depends on the choice of representative within the m-equivalence class of a. More precisely, we even have:

2.1.1. Proposition 1. For every $a\in\mathscr{A}$ there exists an $\tilde a\in\mathscr{A}$ with

$$d_{\tilde a}^*\equiv 0 \text{ and } \tilde a(x)=a(x) \text{ for a.e. } x\in X.$$

Proof. Choose a countable dense set $\{x_n\}_{n\in\mathbb{N}}\subset X$ and a countable set $\{\gamma_{kl}\}_{k,l\in\mathbb{N}}$ of Lipschitz curves $\gamma_{kl}:[0,1]\to X$ connecting $x_k=\gamma_{kl}(0)$ and $x_l=\gamma_{kl}(1)$. Put

$$X_0=\bigcup_{k,l\in\mathbb{N}}\gamma_{kl}([0,1]) \quad\text{and}\quad \tilde a_{ij}(x)=1_{X\setminus X_0}(x)\cdot a_{ij}(x)$$

for $x\in X$ and i, $j\in\{1,\cdots,n\}$. Then $\tilde a(x)=a(x)$ for a.e. $x\in X$ and $\tilde a(x)=0$ for $x\in X_0$. Hence, $L_{\tilde a}(\gamma_{kl})=0$ and thus $d_{\tilde a}^*(x_k,x_l)=0$ for all k, $l\in\mathbb{N}$. But this already implies $d_{\tilde a}^*(x,y)=0$ for all x, $y\in X$ since $\{x_n\}_{n\in\mathbb{N}}$ is dense in X w.r.t. the d_a-topology (which obviously coincides with the $d_{\tilde a}$-topology and is therefore finer than the $d_{\tilde a}^*$-topology).

Of course, the Dirichlet form (and therefore also the diffusion process as well as the heat kernel) associated with the diffusion matrix a only depends on the m-equivalence of class a. Hence, in the general case of discontinuous diffusion matrices a, the path metric d_a^* is of no use. The following alternative characterization of the intrinsic metric in terms of paths is due to De Cecco–Palmieri [6, 7].

2.1.2. Proposition 2. For any $a\in\mathscr{A}$ and all x, $y\in X$

$$d_a(x,y)=\sup_{\substack{X_0\subset X\\ m(X_0)=0}}\ \inf_{\gamma\in\Gamma(x,y,X_0)} L_a(\gamma) \tag{2}$$

where $\Gamma(x,y,X_0)$ denotes the set of all Lipschitz paths $\gamma:[0,1]\to \mathrm{X}$ with $\gamma(0)=x$, $\gamma(1)=y$, and $\gamma(t)\notin X_0$ for a.e. $t\in[0,1]$.

3. THE INTRINSIC METRIC AND THE SMALL TIME ASYMPTOTIC

Let us consider the *diffusion process* (X_t, P_a^x) associated with the Dirichlet form $(\epsilon_a, \mathscr{D}(\epsilon_a))$. Formally, its generator is given by the *divergence form operator*

$$A_a f = \frac{1}{\varphi}\langle \nabla, \varphi a \nabla f\rangle$$

on $L^2(X, m(\mathrm{d}x)) = L^2(X, \varphi(x)\mathrm{d}x)$. The heat kernel $p_a(t,x,y)$ is defined as the density of the transition semigroup

$$P_a^x[X_t \in \mathrm{d}y] = p_a(t,x,y) m(\mathrm{d}y)$$

or, equivalently, as the fundamental solution of the parabolic equation $(A_a - \partial/\partial t)u = 0$.

The following fundamental relation between the heat kernel and the intrinsic metric was derived by Norris [1].

3.1.1. Theorem 1. For any $a \in \mathscr{A}$ and all x, $y \in X$

$$d_a(x,y)^2 = -\lim_{t\to 0} 2t \cdot \log p_a(t,x,y). \tag{3}$$

This result is the solution to a famous problem in analysis which was open since 30 years. Namely, Varadhan [8] proved

$$d_a(x,y)^2 = -\lim_{t\to 0} 2t \cdot \log \tilde{p}_a(t,x,y) \tag{4}$$

for any $a \in \mathscr{A}$ and all x, $y \in X$. Here $\mathscr{A}$ and d_a is defined as before but $\tilde{p}_a(t,x,y)$ is associated with the *non-divergence* form operator

$$\tilde{A}_a f = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Aronson [9] derived uniform upper and lower Gaussian estimates for the heat kernel $p_a(t,x,y)$ associated with a *divergence* form operator $\langle \nabla, a\nabla\rangle$ on $L^2(X, \mathrm{d}x)$ with $a \in \mathscr{A}$. This result can easily be extended to divergence form operators $A_a = 1/\varphi \langle \nabla, \varphi a \nabla\rangle$ on $L^2(X, \varphi(x)\mathrm{d}x)$ as considered before. These estimates immediately imply

$$\frac{1}{C} \cdot \limsup_{t\to 0} 2t \log p_a(t,x,y) \le -d_a^2(x,y) \le C \cdot \liminf_{t\to 0} 2t \log p_a(t,x,y) \tag{5}$$

uniformly in x, y (with d_a as before). Since that time, it has been an open question whether the rough estimate (5) can be improved to get the precise assertion (3). Many people have tried to prove (3) for the described type of divergence form operators with non-smooth coefficients. Davies [10] succeeded in proving the precise upper estimate

$$\limsup_{t\to 0} 2t \log p_a(t,x,y) \le -d_a^2(x,y)$$

for any divergence form operator as described above. His method also applies to more general operators, e.g. to the class of selfadjoint subelliptic operators studied by Fefferman and Phong. In particular, this includes the class of selfadjoint Hörmander-type operators (= sum of squares of $\mathscr{C}^\infty$-vector fields). For the latter, Leandré [11] derived the precise asymptotic (3). However, concerning the precise lower estimate for the described class of divergence form operators with non-smooth coefficients, up to now only partial results have been obtained. The precise asymptotic (3) was obtained for certain subclasses:

—by Norris and Stroock [12], under the restriction that the a_{ij} are continuous and $\varphi \equiv 1$;
—by Zheng [13, 14], in the case $a_{ij} \equiv id$;
—by Norris [5], in the case $\varphi \equiv 1$.

Now, finally, Norris [1] has presented the proof of the precise asymptotic (3) for general divergence form operators with measurable coefficients a and φ.

4. NON-UNIQUENESS

The previous Theorem 1 indicates that the intrinsic metric associated with a Dirichlet form is the appropriate quantity in order to describe the short time asymptotic of the associated diffusion process or heat kernel. Moreover, in many other estimates for heat kernels, Green functions, capacities, hitting probabilities, harmonic functions etc. the intrinsic metric plays a crucial role. See, for example, Sturm [3, 4, 15–17]. Therefore, the question arises: *Is the Dirichlet form determined uniquely by its intrinsic metric*? Or, in other words: *Is the diffusion process determined uniquely by its small time asymptotic*?

The answer is NO. For any symmetric elliptic diffusion there exists another one with the same small time asymptotic but with strictly smaller diffusion coefficients. More precisely, our main result states:

4.1.1. Theorem 2. For every $a \in \mathscr{A}$ there exists an $\tilde{a} \in \mathscr{A}$ with

$$d_{\tilde{a}} \equiv d_a \text{ and } \tilde{a}(x) < a(x) \text{ for all } x \in X$$

(in the sense of forms, i.e. $\langle \xi, \tilde{a}(x)\xi\rangle < \langle \xi, a(x)\xi\rangle$ for all $\xi \in \mathbb{R}^n\setminus\{0\}$). In particular, for any $\epsilon > 0$ and any continuous function $\delta: X \to]0,1[$ one can choose

$$\tilde{a}_{ij}(x) = \Psi(x)\cdot a_{ij}(x)$$

with a measurable function $\Psi: X \to]0,1[$ satisfying $\Psi \geq \delta$ on X and

$$m(\{\Psi > \delta\}) \leq \epsilon.$$

Proof. Let $\delta \in \mathscr{C}(X)$, $\epsilon > 0$ and $\eta > 0$ be given. Choose a countable dense set $\{x_n\}_{n \in \mathbb{N}} \subset X$ and a countable set $\{\gamma_{kl}\}_{k,l \in \mathbb{N}}$ of Lipschitz maps $\gamma_{kl}:[0,1] \to X$ connecting $x_k = \gamma_{kl}(0)$ and $x_l = \gamma_{kl}(1)$ and having length $L_a(\gamma_{kl}) \leq (1+\eta)\cdot d_a(x_k, x_l)$. Note that $m(\gamma_{kl}([0,1])) = 0$. Now choose, for each $k, l \in \mathbb{N}$, a continuous function $\psi_{kl}: X \to [0,1]$ with $\psi_{kl} = 1$ on $\gamma_{kl}([0,1])$ and

$$m(\{\psi_{kl} > \alpha\}) \leq (1-\alpha)\cdot 2^{-k-l}\cdot\epsilon$$

for any $\alpha \in [0,1]$. Then define

$$\Psi_0 = \sup_{k,l} \psi_{kl}$$

which is a lower semicontinuous function $X \to [0,1]$ satisfying $m(\{\psi > \alpha\}) \leq (1-\alpha)\cdot\epsilon$ for any $\alpha \in [0,1]$, in particular, $m(\{\psi > 0\}) \leq \epsilon$ and $m(\{\psi \geq 1\}) = 0$. Finally, put $\Psi_\delta = \delta + (1-\delta)\cdot\Psi_0$ as well as $\Psi_\delta^* = 1_{\{\Psi_0 < 1\}}\cdot\Psi_\delta$. Then obviously, the function $\Psi = \Psi_\delta^*: X \to]0,1[$ is measurable and satisfies $\Psi \geq \delta$ on X as well as $m(\{\Psi > \delta\} \leq \epsilon$. It is also obvious that $\tilde{a} = \Psi\cdot a$ belongs to $\mathscr{A}$.

The main question is: why is $d_{\tilde{a}} \equiv d_a$? Note that of course $\tilde{a} = \Psi_\delta^*\cdot a$ and $\Psi_\delta\cdot a$ lead to the same metric $d_{\tilde{a}}$. Let us consider two points x_k, x_l. The continuity of ψ_{kl} and δ implies that

$$U_{kl} = \left\{\psi_{kl} > \frac{1}{1-\delta}\left(\frac{1}{(1+\eta)^2} - \delta\right)\right\}$$

is an open neighborhood of $\gamma_{kl}([0,1])$. On this open set, $a^* \geq 1/(1+\eta)^2 \cdot a$ (in the sense of forms). Hence,

$$d_a^*(x_k,x_l) \leq L_a^*(\gamma_{kl}) \leq (1+\eta) \cdot L_a(\gamma_{kl}) \leq (1+\eta)^2 \cdot d_a(x_k,x_l).$$

But this already implies $d_a^*(x,y) \leq (1+\eta)^2 \cdot d_a(x,y)$ for all x, $y \in X$ since $\{x_n\}_{n\in\mathbb{N}}$ is dense in X w.r.t. the d_a-topology (which obviously coincides with the d_a^*-topology). In order to finish, it suffices to consider $\eta \to 0$.

Remarks.

1. Let us give an intuitive description of the previous construction. Assume for simplicity that $a_{ij}(x) = \delta_{ij}$. The diffusion process associated with this diffusion matrix is just the *Brownian motion* on $X = \mathbb{R}^n$. Now let us consider the diffusion process associated with the diffusion matrix $\tilde{a}_{ij}(x) = \Psi_\delta \cdot \delta_{ij}$. This process can move *fast* (namely, as fast as Brownian motion) on a small, *fractal like* set $X_1 = \{\Psi_0 = 1\}$ of measure 0 which connects all points of a given dense subset $\{x_n\}_{n\in\mathbb{N}} \subset X$. On the other hand, the process moves *slowly* (namely, with velocity being δ times the velocity of Brownian motion where typically $\delta \ll 1$) on a *large* set $\{\Psi_0 = 0\}$ whose complement has measure $\epsilon \ll 1$.

   The technically important point is, that the diffusion moves *sufficiently fast* in a (sufficiently small) neighborhood of this fractal like set X_1. More precisely, for any $\alpha < 1$ (and $\epsilon > 0$) there exists an open neighborhood X_α of X_1 (with measure $m(X_\alpha) \leq \epsilon$) such that the diffusion process moves on X_α at least with velocity α times the velocity of Brownian motion.
2. The construction of the above diffusion matrix $\tilde{a} = \Psi \cdot a$ was inspired by a similar construction of Davies (1996, private communication). He used it to give an example of a diffusion with $d_a \equiv \infty$ but with $a_{ij} \in L^p(X, m)$ for all $p < \infty$.
3. Assume that the coefficients a_{ij} are smooth. Then in the definition (1) of the intrinsic metric d_a the set $\mathscr{C}(X) \cap \mathscr{D}_{\text{loc}}(\epsilon)$ obviously may be replaced by $\mathscr{C}^1(X)$, i.e.

$$d_a(x,y) = \sup\{f(x) - f(y) : f \in \mathscr{C}^1(X), \langle \nabla f, a \nabla f \rangle \leq 1 \text{ a.e. on } X\}. \tag{6}$$

However, this equality will *not be true in general*! This was observed by Zheng [13]. In our context, a simple example is given by $\tilde{a} = \Psi \cdot a$ with a function Ψ as in Theorem 2.

5. CONTINUOUS COEFFICIENTS

Now let us consider the set $\mathscr{A}_c$ consisting of those $a \in \mathscr{A}$ for which $a_{ij}: X \to \mathbb{R}$ is continuous for each $i, j = 1, \ldots, n$. From Proposition 2 one immediately obtains.

5.1.1. Proposition 3. For any $a \in \mathscr{A}_c d_a \equiv d_a^*$.

The important point for us is that the equality of d_a and d_a^* allows to reconstruct the diffusion matrix a from the metric d_a.

5.1.2. Proposition 4. For any $a \in \mathscr{A}_c$, any $x \in X$ and any $\xi \in \mathbb{N}^n$ with $|\xi| = 1$:

$$\langle \xi, \mathrm{a}(\mathrm{x})\xi \rangle \geq \limsup_{t\to 0} \left(\frac{\mathrm{d}(x, x+t\xi)}{t}\right)^{-2} \geq \liminf_{t\to 0} \left(\frac{\mathrm{d}(x, x+t\xi)}{t}\right)^{-2} \geq \langle \xi, a^{-1}(x)\xi \rangle^{-1}$$

with equality

$$\langle \xi, a(x)\xi \rangle = \lim_{t\to 0} \left(\frac{\mathrm{d}(x, x+t\xi)}{t}\right)^{-2}$$

if ξ is an eigenvector of $a(x)$. In particular, $a \in \mathscr{A}_c$ is determined uniquely by d_a.

Proof. Fix $x \in X$, $\xi \in \mathbb{R}^n \mathbb{N}^n$ and a neighborhood $B \subset X$ of x. From the definition (1) of the intrinsic metric d_a we deduce that

$$\mathrm{d}(x,x+t\xi) \geq t \cdot \left(\sup_{y \in B} \langle \xi, a(y)\xi \rangle\right)^{-1/2}$$

for sufficiently small $t > 0$. (Namely, we can choose $u{:}z \to (\sup_{y \in B}\langle \xi, a(y)\xi\rangle)^{-1/2} \cdot \langle \xi, z\rangle$.)

On the other hand, Corollary 1 and the definition of the path metric d_a^* imply

$$\mathrm{d}(x,x+t\xi) \leq t \cdot \left(\sup_{y \in B} \langle \xi, a^{-1}(y)\xi \rangle\right)^{1/2}$$

for sufficiently small $t > 0$. (Namely, we can choose $\gamma{:}s \to x + st\xi$.) Assuming that a is continuous this already yields the claim.

For details refer to Sturm [18, 19].

REFERENCES

1. Norris, J. R., Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds. In press.
2. Fukushima, M., Oshima, Y. and Takeda, M., *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter, Berlin, 1994.
3. Sturm, K. T., Analysis on local Dirichlet spaces—I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math*, 1994, **456,** 173–196.
4. Sturm, K. T., On the geometry defined by Dirichlet forms. In *Seminar on Stochastic Analysis, Random Fields and Applications*, Ascona, 1993 (eds E. Bolthausen *et al.*). Birkhäuser, Berlin, 1995, pp. 231–242.
5. Norris, J. R., Small time asymptotics for heat kernels with measurable coefficients. *C.R. Acad. Sci. Paris, Serie I*, 1996, **322,** 339–344.
6. De Cecco, G. and Palmieri, G., Integral distance on a Lipschitz Riemannian manifold. *Math. Z.*, 1991, **207,** 223–243.
7. De Cecco, G. and Palmieri, G., LIP manifolds—from metric to Finslerian structures. *Math. Z.*, 1995, **218,** 223–237.
8. Varadhan, S. R. S., On the behaviour of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure Appl. Math.*, 1967, **20,** 431–455.
9. Aronson, D. G., Non-negative solutions of linear parabolic equations. *Ann. Sci. Norm. Sup. Pisa*, 1967, **22,** 607–694.
10. Davies, E. B., Explicit constants for gaussian upper bounds on heat kernels. *Am. J. Math.*, 1987, **109,** 319–334.
11. Léandre, R., Minoration en temps petit de la densité d'une diffusion dégénérée. *J. Funct. Anal.*, 1987, **74,** 399–414.
12. Norris, J. R. and Stroock, D. W., Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients. *Proc. Lond. Math. Soc.*, 1991, **62,** 373–402.
13. Zheng, W., A large deviation result for a class of Dirichlet processes. *Prob. Theory Related Fields*, 1995, **101,** 237–249.
14. Zheng, W., Diffusion processes on a Lipschitz Riemannian manifold and their applications. *Proc. of Symposia in Pure Mathematics*, 1995, **57,** 373–381.
15. Sturm, K. T., Analysis on local Dirichlet spaces—II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.*, 1995, **32,** 275–312.
16. Sturm, K. T., Sharp estimates for capacities and applications to symmetric diffusions. *Prob. Theory Related Fields*, 1995, **102,** 73–89.
17. Sturm, K. T., Analysis on local Dirichlet spaces—III. The parabolic Harnack inequality. *J. Math. Pure Appl.*, 1996, **75,** 273–297.
18. Sturm, K. T., How to construct diffusion processes on metric spaces. *Potential Analysis*. In press.
19. Sturm, K. T., Diffusion processes and heat kernels on length spaces. *Ann. Prob. In press.*