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ANALYSIS ON LOCAL DIRICHLET SPACES III. THE PARABOLIC HARNACK INEQUALITY

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ABSTRACT. – In the context of local Dirichlet spaces we prove that the parabolic Harnack inequality holds true if and only if the doubling property and the Poincaré inequality hold true. An important observation is that (under the doubling property) the weak Poincaré already implies the strong Poincaré inequality and also the Sobolev inequality.

Using the parabolic Harnack inequality we deduce upper and lower Gaussian estimates for the heat kernel associated with the Dirichlet form.

These results depend on the fact that the Dirichlet form itself defines in an intrinsic way a metric on the state space and that this metric allows to construct geodesics on the state space.

1. The Dirichlet space and the intrinsic metric

A) The Dirichlet space $(\mathcal{E}, \mathcal{F})$

The basic object for the sequel is a fixed regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{F} = \mathcal{F}(X)$ on a real Hilbert space $L^2(X, m)$ with norm $\|u\| = \left(\int_X u^2\, dm\right)^{1/2}$. $\mathcal{F}$ is again a real Hilbert space with norm $\|u\|_{\mathcal{F}} := \sqrt{\mathcal{E}(u, u) + \|u\|^2}$.

The underlying topological space X is a locally compact separable Hausdorff space and m is a positive Radon measure with supp $[m] = X$. The initial Dirichlet form $\mathcal{E}$ is always assumed to be *symmetric* (i.e. $\mathcal{E}(u, v) = \mathcal{E}(v, u)$) and *strongly local* (i.e. $\mathcal{E}(u, v) = 0$ whenever $u \in \mathcal{F}$ is constant on a neighborhood of the support of $v \in \mathcal{F}$ or, in other words, $\mathcal{E}$ has no killing measure and no jumping measure). The selfadjoint negative semidefinite operator associated with the initial form $\mathcal{E}$ is denoted by L.

B) The energy measure Γ

Any such form can be written as

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v)$$

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where Γ is a positive semidefinite, symmetric bilinear form on $\mathcal{F}$ with values in the signed Radon measures on X (the so-called energy measure). It can be defined by the formulae

$$\int_X \phi\, d\Gamma(u, u) = \mathcal{E}(u, \phi u) - \frac{1}{2}\mathcal{E}(u^2, \phi)$$
$$= \lim_{t \to 0} \frac{1}{2t} \int_X \int_X \phi(x) \cdot [u(x) - u(y)]^2\, T_t(x, dy)\, m(dx)$$

for every $u \in \mathcal{F}(X) \cap L^\infty(X, m)$ and every $\phi \in \mathcal{F}(X) \cap \mathcal{C}_0(X)$. Since $\mathcal{E}$ is assumed to be strongly local, the energy measure Γ is local and satisfies the Leibniz rule as well as the chain rule, *cf.* [Fu], [Le], [M], [St1]. As usual we extend the quadratic forms $u \mapsto \mathcal{E}(u, u)$ and $u \mapsto \Gamma(u, u)$ to the whole spaces $L^2(X, m)$ resp. $L^2_{\text{loc}}(X, m)$ in such a way that $\mathcal{F}(X) = \{u \in L^2(X, m) : \mathcal{E}(u, u) < \infty\}$ and $\mathcal{F}_{\text{loc}}(X) = \{u \in L^2_{\text{loc}}(X, m) : \Gamma(u, u)$ is a Radon measure$\}$.

C) The intrinsic metric ρ

The energy measure Γ defines in an intrinsic way a pseudo metric ρ on X by

$$\rho(x, y) = \sup\{u(x) - u(y) : u \in \mathcal{F}_{\text{loc}}(X) \cap \mathcal{C}(X),\ \Gamma(u, u) \le m \text{ on } X\}, \tag{1.1}$$

called intrinsic metric or Carathéodory metric (*cf.* [BM1, 2], [Da], [VSC]). The condition $\Gamma(u, u) \le m$ in (1.1) means that the energy measure $\Gamma(u, u)$ is absolutely continuous w.r.t. the reference measure m with Radon-Nikodym derivative $\frac{d}{dm}\Gamma(u, u) \le 1$. The density $\frac{d}{dm}\Gamma(u, u)(z)$ should be interpreted as the square of the (length of the) gradient of u at $z \in X$. In general, ρ may be degenerate (*i.e.* $\rho(x, y) = \infty$ or $\rho(x, y) = 0$ for some $x \neq y$).

D) Strong regularity

In addition to the assumptions from section A) we assume from now on that the Dirichlet form $\mathcal{E}$ is *strongly regular* in the sense of the following

Definition. – A strongly local, symmetric Dirichlet form $\mathcal{E}$ with domain $\mathcal{F} \subset L^2(X, m)$ is called *strongly regular* if it is regular and if ρ (defined by (1.1)) is a metric on X whose topology coincides with the original one.

The strong regularity in particular implies that ρ is non-degenerate, that X is connected and that for any $y \in X$ the function $x \mapsto \rho(x, y)$ is continuous on X. Note that due to the strong regularity any ball $B_r(x) = \{y \in X : \rho(x, y) < r\}$ is connected and its boundary always coincides with the sphere $S_r(x) = \{y \in X : \rho(x, y) = r\}$ ([St3], Proposition 1). Moreover, for fixed $x \in X$ and *sufficiently small* $r > 0$ the closed balls $\overline{B}_r(x)$ are compact and thus complete. Here and in the sequel, completeness always means completeness with respect to the metric ρ. That is, a subset $Y \subset X$ is called *complete* if and only if the metric space (Y, ρ) is complete.

LEMMA 1.1. – (i) *A closed ball* $\overline{B}_r(x) \subset X$ *is complete if and only if it is compact.*

(ii) *The whole space X is complete if and only if all closed balls* $\overline{B}_r(x)$ *are compact in X.*

Proof. – Assertion (ii) was already proved as Theorem 2 in [St3]. The first assertion can be proved with the same argument using the following Lemma 1.3 (which is the appropriate generalization of Theorem 1 in [St3]). □

LEMMA 1.2. – (i) *If a closed ball* $\overline{B}_r(x) \subset X$ *is complete then any point* $y \in \overline{B}_r(x)$ *can be joined with the center x by a minimal geodesic in* $\overline{B}_r(x)$. *That is, there exists a continuous map* $\gamma : [0, 1] \to Y$ *with* $\gamma(0) = x$, $\gamma(1) = y$ *and*

$$\rho(\gamma(r), \gamma(t)) = \rho(\gamma(r), \gamma(s)) + \rho(\gamma(s), \gamma(t)) \tag{1.2}$$

for all $0 \le r < s < t \le 1$.

(ii) *If the whole space X is complete then* (X, ρ) *is a geodesic space,* i.e. *any two points* $x, y \in X$ *can be joined by a minimal geodesic in X.*

Proof. – [St3], Theorem 1 and Remark thereafter.

Note, however, that assertion (i) of the above Lemma does *not* imply that the complete ball $\overline{B}_r(x)$ is *convex*, that is, it does *not imply* that any two points $y, z \in \overline{B}_r(x)$ can be joined by a minimal geodesic in $\overline{B}_r(x)$.

The proof of both preceding Lemmata essentially depends on the following basic property of the distance function which was derived as Lemma 1.1 in [St1].

LEMMA 1.3. – *For every* $y \in X$ *the distance function* $\rho_y : x \mapsto \rho(x, y)$ *satisfies* $\rho_y \in \mathcal{F}_{\text{loc}}(X) \cap \mathcal{C}(X)$ *and*

$$\Gamma(\rho_y, \rho_y) \le m. \tag{1.3}$$

Hence, the distance function ρ_y can be used to construct cut-off functions on intrinsic balls $\overline{B}_r(y)$ of the form $\rho_{y,r} : x \mapsto (r - \rho(x, y))_+$.

2. The Poincaré inequality

Throughout this chapter, we fix an arbitrary subset $Y \subset X$. We state and discuss several properties of the Dirichlet form $\mathcal{E}$ on this set Y.

A) Completeness and doubling properties

PROPERTY (Ia) (Completeness property). – *For all balls* $B_{2r}(x) \subset Y$ *the closed balls* $\overline{B}_r(x)$ *are complete (or, equivalently, compact).*

Of course, Property (Ia) is satisfied if $\overline{Y}$ is complete. Note that due to the strong regularity, each point $x \in X$ has a neighborhood $Y = Y(x)$ with this property, and that even each set $Y \subset X$ has this property if X is complete. Let us also mention that (Ia) obviously implies that a closed ball $\overline{B}_r(x)$ is complete as soon as $B_{\gamma r}(x) \subset Y$ for some $\gamma > 1$.

PROPERTY (Ib) (Doubling property). – *There exists a constant* $N = N(Y)$ *such that for all balls* $B_{2r}(x) \subset Y$

$$m(B_{2r}(x)) \leq 2^N \cdot m(B_r(x)). \tag{2.1}$$

Note that (2.1) implies $m(B_{r'}(x)) \leq (2r'/r)^N \cdot m(B_r(x))$ for all $B_r(x) \subset B_{r'}(x) \subset Y$ and $m(B_{r'}(x')) \leq (4r'/r)^N \cdot m(B_r(x))$ for all $B_r(x) \subset B_{r'}(x') \subset Y$.

The number N plays the role of the dimension of the space Y. Note, however, that it may be a fractional number. Let us mention that without restriction this number N in (2.1) can always be chosen to satisfy $N > 2$.

With properties (Ia) and (Ib) at hand, one is already in a position to estimate the following spectral bounds

$$\lambda^{\mathrm{Dir}}(B) = \inf\left\{\frac{\mathcal{E}(u, u)}{\|u\|^2} : u \in \mathcal{F},\ u \neq 0,\ \tilde{u} = 0 \text{ q.e. on } X\backslash B\right\} \tag{2.2}$$

and

$$\lambda^{\mathrm{Neu}}(B) = \inf\left\{\frac{\int_B d\Gamma(u, u)}{\int_B u^2\, dm} : u \in \mathcal{F},\ u \neq 0,\ \int_B u\, dm = 0\right\} \tag{2.3}$$

for the operator $-L$ with suitable boundary conditions on ∂B where B is some ball in X. $\lambda^{\mathrm{Dir}}(B)$ is the bottom of the spectrum of the operator $-L$ on $L^2(B, m)$ with "Dirichlet" boundary conditions on ∂B and $\lambda^{\mathrm{Neu}}(B)$ is the spectral gap (between zero and the first non-zero spectral value) for the operator $-L$ on $L^2(\overline{B}, m)$ with "Neumann" boundary conditions on ∂B.

PROPOSITION 2.1. – *Assume (Ia) and (Ib). Then for all balls* $B = B_r(x) \subset Y$ *of radius* r

$$\lambda^{\mathrm{Dir}}(B) \leq \frac{2^{N+2}}{r^2} \tag{2.4}$$

and if $\partial B_r(x) \neq \emptyset$ *and* $B_{3r}(x) \subset Y$ *then*

$$\lambda^{\mathrm{Neu}}(B) \leq \frac{8^{N+2}}{r^2}. \tag{2.5}$$

Proof. – In order to estimate $\lambda^{\mathrm{Dir}}(B)$ consider the function $u(y) = \rho(y, X\backslash B_r(x))$. From Lemma 1.3 it follows that $\Gamma(u, u) \leq 1_{B_r(x)}\, m$ and thus $\mathcal{E}(u, u) \leq m(B_r(x))$. On the other hand, $\|u\|^2 \geq \frac{r^2}{4} \cdot m(B_{r/2}(x)) \geq 2^{-2-N} \cdot r^2 \cdot m(B_r(x))$. This yields the first claim.

For the proof of the second claim, we consider the function $v = u - C$ where u as above and $C = \int_{B_r(x)} u\, dm / m(B_r(x))$. Of course, $\int_{B_r(x)} d\Gamma(v, v) = \mathcal{E}(u, u) \leq m(B_r(x))$.

In order to estimate $\|u - C\|^2$ we distinguish two cases: either $C < 3/4\,r$ or $C \geq 3/4\,r$. In the first case,

$$\int_{B_r(x)} |u - C|^2\, dm \geq \int_{B_{r/8}(x)} \left(\frac{r}{8}\right)^2 dm \geq 8^{-2-N} \cdot r^2 \cdot m(B_r(x)).$$

In the second case, we make use of the assumption $\partial B_r(x) \neq \emptyset$. It implies that there exists a point y with $\rho(x, y) = 3/4\,r$. Using the assumption $C \geq 3/4\,r$ we get

$$\begin{aligned}\int_{B_r(x)} |u - C|^2\, dm &\geq \int_{B_r(x)\setminus B_{r/2}(x)} \left(\frac{r}{4}\right)^2 dm \geq 2^{-4} \cdot r^2 \cdot m(B_{r/4}(y))\\ &\geq 8^{-2-N} \cdot r^2 \cdot m(B_{2r}(y)) \geq 8^{-2-N} \cdot r^2 \cdot m(B_r(x)). \quad \square\end{aligned}$$

Property (Ib) implies that the metric space (Y, ρ) is a homogeneous space in the sense of Coifman and Weiss [CW] which in turn implies that several covering properties hold true. The most important for us is the Whitney covering property below. We use the following notations. For a ball $B = B_r(x)$, we denote $B' = B_{2r}(x)$, $B'' = B_{4r}(x)$ and $B^* = B_{10\,r}(x)$ and we write $\rho(B)$ for the radius r of B.

LEMMA 2.2. – *Assume (Ia) and (Ib) and fix $Z = B_R(z) \subset Y$. There is a pairwise disjoint family $\mathcal{G}$ of balls ("Whitney family") and a constant $f(N)$ depending only on the doubling constant N in* (2.1) *such that*

(i) $Z = \bigcup_{B \in \mathcal{G}} B'$;

(ii) $10^2\,\rho(B) \leq \rho(B, X\setminus Z) \leq 10^3\,\rho(B)$ *for all* $B \in \mathcal{G}$

(iii) $\sharp\{B \in \mathcal{G} : x \in B^*\} \leq f(N)$ *uniformly for all* $x \in Z$.

Here and below, $\sharp\{\cdot\}$ denotes the number of elements in the set $\{\cdot\}$. The proof of this Lemma is well known and straightforward (*cf.* [Je]). More sophisticated is the estimate for the number of balls in this Whitney family which are needed to join a point close to the boundary of $B_R(z)$ with the center z. To this end, we recall that by assumption (Ia) each ball $B_r(z)$ with $r < R$ is relatively compact which allows to connect points in Z with the center z by geodesics.

For $B \in \mathcal{G}$, define γ_B as a minimal geodesic from the center of B to the center z of $Z = B_R(z)$ and denote the graph of γ_B also by γ_B. The length of this path γ_B is then $\leq R$ and the whole path lies inside of Z (*i.e.* $\gamma_B \subset Z$). Of course, this path need not to be unique but exactly one will be associated to every ball $B \in \mathcal{G}$.

Following [Je] and [SaS], we denote $\mathcal{G}(B) = \{A \in \mathcal{G} : A' \cap \gamma_B \neq \emptyset\}$ for any $B \in \mathcal{G}$ and $\mathcal{G}^*(B) = \{A \in \mathcal{G} : B \in \mathcal{G}(A)\}$.

LEMMA 2.3. – *Assume (Ia) and (Ib) and let B belong to $\mathcal{G}$. Then:*

(i) $\rho(A) \geq 10^{-2}\,\rho(B)$ *for every* $A \in \mathcal{G}(B)$;

(ii) *there exists a constant $f(N)$ only depending on the doubling constant N such that*

$$\frac{\rho(B)^2}{m(B)} \sum_{A \in \mathcal{G}^*(B)} \sharp\mathcal{G}(A) \cdot m(A) \leq f(N) \cdot R^2.$$

The proof of this Lemma is entirely analogous to the original proof of Jerison ([Je], Lemmata 5.6-5.9, *cf.* also [SaS], [Lu]). Note, however, that in this argument it is not sufficient to have *any* metric ρ which satisfies Properties (Ia) and (Ib) but it is necessary to have a metric which allows to construct *geodesics* (*cf.* our Lemma 1.2).

B) Weak and strong Poincaré inequalities

The basic quantitative property in the sequel will be the Poincaré or spectral gap inequality. There are various forms of this inequality which will be shown to be equivalent to each other.

PROPERTY (Ic) (Weak Poincaré inequality). – *There exists a constant $C_P = C_P(Y)$ such that for all balls $B_{2r}(x) \subset Y$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \, dm \leq C_P \cdot r^2 \int_{B_{2r}(x)} d\Gamma(u, u) \tag{2.6}$$

for all $u \in \mathcal{F}(X)$ where $u_{x,r} = \dfrac{1}{m(B_r(x))} \displaystyle\int_{B_r(x)} u \, dm.$

This property will be compared with the following stronger property (where the ball B_{2r} on the RHS of (2.6) is now replaced by the ball B_r).

PROPERTY (Ic*) (Strong Poincaré inequality). – *There exists a constant $C_P^* = C_P^*(Y)$ such that for all balls $B_r(x) \subset Y$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \, dm \leq C_P^* \cdot r^2 \int_{B_r(x)} d\Gamma(u, u) \tag{2.7}$$

for all $u \in \mathcal{F}(X)$ where $u_{x,r} = \dfrac{1}{m(B_r(x))} \displaystyle\int_{B_r(x)} u \, dm.$

Using the definition (2.3), inequality (2.7) is equivalent to

$$\lambda^{\mathrm{Neu}}(B_r(x)) \geq \frac{1}{C_P^* \cdot r^2}. \tag{2.8}$$

This should be compared with the upper bound (2.5) which is of the same order. Note that the set of assumptions used in [BM2] also contains an analogous lower bound $\lambda^{\mathrm{Dir}}(B_r(x)) \geq \dfrac{1}{C \cdot r^2}$. In our context, this will not be required and will in general not be true.

THEOREM 2.4. – *Assume that (Ia) and (Ib) hold true. Then the weak Poincaré inequality (Ic) holds true if and only if the strong Poincaré inequality (Ic*) holds true.*

The strong Poincaré constant C_P^* can be chosen to be $f(N) \cdot C_P$ where $f(N)$ is a constant only depending on the doubling constant N from (2.1). On the other hand, the weak Poincaré constant C_P can of course be chosen to be C_P^*.

Proof. – Having at hand the Whitney covering from Lemma 2.2 and the estimates from Lemma 2.3, the proof of Jerison [Je] applies without changes (*cf.* also [SaS], [Lu]). □

COROLLARY 2.5 (Weighted Poincaré inequality). – *Assume that (Ia) and (Ib) hold true. Then the weak Poincaré inequality (Ic) holds true if and only if there exists a constant $C'_P = C'_P(Y)$ such that for all balls $B_r(x) \subset Y$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \phi^2 \, dm \leq C'_P \cdot r^2 \int_{B_r(x)} \phi^2 \, d\Gamma(u, u) \tag{2.9}$$

for all $u \in \mathcal{F}(X)$ where $\phi = (1 - \rho(x, \cdot)/r) \vee 0$ and $u_{x,r} = \int_{B_r(x)} u \cdot \phi^2 \, dm / \int_{B_r(x)} \phi^2 \, dm$. The constant C'_P can be chosen to be $f(N) \cdot C_P$ where $f(N)$ is a constant only depending on the doubling constant N and, conversely, C_P can be chosen to be $f'(N) \cdot C'_P$.

Proof. – The assertion can be proven either directly starting with the weak Poincaré inequality (Ic) and using the arguments of [SaS] or starting already with the strong Poincaré inequality (Ic*) and observing the following facts:

(i) the measure $\phi^2 \cdot m$ is comparable with the measure $\sum_{k=1}^{\infty} 4^{-k} \cdot 1_{B(k)} \cdot m$ where $B(k) = B_{(1-2^{-k})r}(x)$;

(ii) $\int_{B(k)} |u - u_1|^2 \, dm \leq f_1(N) \cdot \left[\int_{B(k)} |u - u_k|^2 \, dm + \int_{B(1)} |u - u_1|^2 \, dm \right]$ where $u_k = \int_{B(k)} u \, dm / m(B(k))$;

(iii) $\int_{B(k)} |u - u_k|^2 \, dm \leq C_P^* \cdot r^2 \int_{B(k)} d\Gamma(u, u)$. Hence,

$$\begin{aligned}
\int_{B(\infty)} |u - u_1|^2 \cdot \phi^2 \, dm &\leq 4 \cdot \sum_{k=1}^{\infty} 4^{-k} \int_{B(k)} |u - u_1|^2 \, dm \\
&\leq f_1(N) \cdot \left[\frac{4}{3} \int_{B(1)} |u - u_1|^2 \, dm + \sum_{k=2}^{\infty} 4^{-k} \int_{B(k)} |u - u_k|^2 \, dm \right] \\
&\leq f_2(N) \cdot C_P^* \cdot r^2 \cdot \sum_{k=1}^{\infty} 4^{-k} \int_{B(k)} d\Gamma(u, u) \leq f_3(N) \cdot C_P^* \cdot r^2 \int_{B(\infty)} \phi^2 \, d\Gamma(u, u). \quad \square
\end{aligned}$$

Remark. – The function $\phi = (1 - \rho(x, \cdot)/r) \vee 0$ in Corollary 2.5 can actually be replaced by a more general weight function ϕ on $B_r(x)$. Sufficient conditions for ϕ can be found in [SaS].

C) Poincaré and Sobolev inequalities

In order to prove an elliptic or parabolic Harnack inequality (e.g. for subelliptic operators on $\mathbb{R}^N$ or for Laplace-Beltrami operators on Riemannian manifolds) using the method of Moser [Mo1, 2], it was general knowledge since quite a long time that it suffices to have a doubling property, a Sobolev inequality and a weighted Poincaré inequality. Only recently, independently Grigor'yan [Gr] and Saloff-Coste [Sa2] could prove that (at least in Riemannian geometry), a doubling property and a Poincaré inequality already imply a Sobolev inequality. The proof of Saloff-Coste carries over to our general situation.

THEOREM 2.6. (Sobolev inequality). – *Assume that (Ia), (Ib) and (Ic) hold true and put* $N^* = \sup\{N, 3\}$. *Then there exists a constant* $C_S = C_S(Y)$ *such that for all balls* $B_{2r}(x) \subset Y$

$$\left(\int_{B_r(x)} |u|^{\frac{2N^*}{N^*-1}} dm\right)^{\frac{N^*-2}{N^*}} \leq C_S \cdot \frac{r^2}{m(B_r(x))^{2/N^*}} \int_{B_r(x)} d\Gamma(u, u) + r^{-2} \cdot u^2\, dm \tag{2.11}$$

for all $u \in \mathcal{F}(X) \cap \mathcal{C}_0(B_r(x))$. *The constant* C_S *can be chosen to be* $f(N) \cdot C_P$ *where* $f(N)$ *is a constant only depending on the doubling constant N.*

Proof. – We follow the proof of Theorem 2.1 of Saloff-Coste [Sa2]. The abstract arguments (Theorem 2.2, Lemma 2.3) carry over line by line. The geometric argument in Lemma 2.4 of [Sa2] follows from the fact that according to the doubling property (Ib) the metric space (X, ρ) is a homogeneous space in the sense of Coifman/Weiss [CW]. In particular, the relevant "Vitali covering property" holds true ([CW], p. 69, *cf.* [FrS], pp. 548-549). □

Remarks. – (i) Using the cut-off functions from Lemma 1.1 it is easy to see (*cf.* section 2.1 B in [St2]) that (2.10) implies that there exists a constant $C_S^* = C_S^*(Y)$ such that for all balls $B_{2r}(x) \subset Y$

$$\left(\int_{B_{(2-\delta)r}(x)} |u|^{\frac{2N^*}{N^*-2}} dm\right)^{\frac{N^*-2}{N^*}} \leq C_S^* \cdot \frac{r^2}{m(B_r(x))^{2/N^*}} \cdot \int_{B_{2r}(x)} d\Gamma(u, u) + (\delta r)^{-2} \cdot u^2\, dm, \tag{2.11}$$

for all $u \in \mathcal{F}_{\text{loc}}(X)$ and all $\delta < 2$.

(ii) The constant N^* in Theorem 2.6 can actully be chosen to be any number $\geq N$ and > 2.

3. The parabolic Harnack inequality

Throughout this chapter, we again fix an arbitrary subset $Y \subset X$.

A) Parabolic equations and one-parameter families of Dirichlet forms

In the sequel we will study the behaviour of local solutions of the parabolic equation $\left(L_t - \frac{\partial}{\partial t}\right) u = 0$ on $\mathbb{R} \times X$. Here $\{L_t\}_{t \in R}$ is a uniformly parabolic operator in the following sense. We assume that for every $t \in \mathbb{R}$ we are given a regular, strongly local and symmetric Dirichlet form $\mathcal{E}_t$ with domain $\mathcal{D}(\mathcal{E}_t) \equiv \mathcal{F}$. The negative semidefinite, selfadjoint operator on $\mathcal{H} = L^2(X, m)$ associated with the Dirichlet form $\mathcal{E}_t$ is denoted by

L_t. That is, $\mathrm{dom}(L_t) \subset \mathcal{F}$ and $-(L_t u, v) = \mathcal{E}_t(u, v)$ for all $u \in \mathrm{dom}(L_t)$ and $v \in \mathcal{F}$. The one-parameter family $\{\mathcal{E}_t\}_{t \in R}$ of these Dirichlet forms is assumed to be *uniformly parabolic* with respect to the initial Dirichlet form $\mathcal{E}$ in the following sense: there exists a constant κ such that for all $u \in \mathcal{F}$ and all $t \in \mathbb{R}$

$$1/\kappa \cdot \mathcal{E}(u, u) \leq \mathcal{E}_t(u, u) \leq \kappa \cdot \mathcal{E}(u, u). \tag{3.1}$$

Let us identify the Hilbert space $\mathcal{H} = L^2(X, m)$ with its own dual and denote the dual of $\mathcal{F}$ by $\mathcal{F}^*$. Then we have

$$\mathcal{F} \subset \mathcal{H} \subset \mathcal{F}^*$$

with continuous and dense embeddings. We shall use the same notation $(\cdot, \cdot)$ for the inner product in $\mathcal{H}$ and for the pairing between $\mathcal{F}^*$ and $\mathcal{F}$.

Let $I =]\sigma, \tau[\subset \mathbb{R}$ be an open interval. We will be concerned with the following Banach spaces (*cf.* [St2]):

• $L^2(I \to \mathcal{F})$ being the Hilbert space of functions of the form $u : I \to \mathcal{F}$, $t \mapsto u_t = u(t, \cdot)$ equipped with the norm $\left(\int_I \|u_t\|_{\mathcal{F}}^2 \, dt\right)^{1/2}$. What one has in mind is that actually $u : (t, x) \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$ is a function of space and time which is regarded as a one-parameter family $(u_t)_{t \in I}$ of functions u_t depending only on space. Note that in this paper u_t *always* denotes the function $x \mapsto u(t, x)$ and *never* the time derivative of u. The latter is always denoted by $\frac{\partial}{\partial t} u$.

• $H^1(I \to \mathcal{F}^*)$ the Sobolev space of functions $u \in L^2(I \to \mathcal{F}^*)$ with distributional time derivative $\frac{\partial}{\partial t} u \in L^2(I \to \mathcal{F}^*)$ equipped with the norm

$$\left(\int_I \|u_t\|_{\mathcal{F}^*}^2 + \left\|\frac{\partial}{\partial t} u_t\right\|_{\mathcal{F}^*}^2 dt\right)^{1/2}.$$

• $\mathcal{F}(I \times X) := L^2(I \to \mathcal{F}) \cap H^1(I \to \mathcal{F}^*)$ being a Hilbert space with norm

$$\|u\|_{\mathcal{F}(I \times X)} = \left(\int_I \|u_t\|_{\mathcal{F}}^2 + \left\|\frac{\partial}{\partial t} u_t\right\|_{\mathcal{F}^*}^2 dt\right)^{1/2}.$$

Let G be an open subset of X, let I be the interval $]\sigma, \tau[\subset \mathbb{R}$ and let Q be the *parabolic cylinder* $I \times G$. Denote the measure $dt \otimes dm$ on $\mathbb{R} \times X$ by $d\underline{m}$. We define $\mathcal{F}_{\mathrm{loc}}(Q)$ to be the set of all $\underline{m}$-measurable functions on Q such that for every relatively compact, open set $G' \subset\subset G$ and every open interval $I' \subset\subset I$ there exists a function $u' \in \mathcal{F}(I \times X)$ with $u = u'$ on $I' \times G'$. We say that a function u belongs to $\mathcal{F}_{\Box}(I \times G)$ if $u \in \mathcal{F}(I \times X)$ and if for a.e. $t \in I$ the function u_t has compact support in G. Note that a function $u \in \mathcal{F}_{\Box}(I \times G)$ only has to vanish on the lateral boundary $I \times \partial G$ but neither on the upper boundary $\{\tau\} \times G$ nor on the lower boundary $\{\sigma\} \times G$.

DEFINITION. – We say that u is a *local solution* of the parabolic equation

$$\left(L_t - \frac{\partial}{\partial t}\right) u = 0 \quad \text{on } Q$$

iff $u \in \mathcal{F}_{\mathrm{loc}}(Q)$ and

$$\int_J \mathcal{E}_t(u_t, \phi_t)\, dt + \int_I \left(\frac{\partial}{\partial t} u_t, \phi_t\right) dt = 0, \tag{3.2}$$

for all $J \subset\subset I$ and all $\phi \in \mathcal{F}_\square(Q)$.

In order to be precise one could call our solutions *local weak solutions.* Note that if for a.e. $t \in \mathbb{R}$ the function $u(t, \cdot)$ is locally in the domain of the operator L_t then (3.2) is satisfied if and only if the functions $L_t u(t, \cdot)$ satisfy $L_t u(t, x) = \frac{\partial u}{\partial t}(t, x)$ for $\underline{m}$-a.e. $(t, x) \in Q$.

B) The parabolic Harnack inequality

The *Harnack inequality* is a uniform estimate for the growth of local solutions of certain operator equations (usually, partial differential equations). The *elliptic* Harnack inequality deals with local solutions $u : x \mapsto u(x)$ of the equation $Lu = 0$ on X; the *parabolic* Harnack inequality with local solutions $u : (t, x) \mapsto u(t, x)$ of the equation $\left(L - \frac{\partial}{\partial t}\right) u = 0$ on $\mathbb{R} \times X$ or of the equation $\left(L_t - \frac{\partial}{\partial t}\right) u = 0$ on $\mathbb{R} \times X$.

PROPERTY (II) $\left(\text{Parabolic Harnack inequality for the operator } L - \frac{\partial}{\partial t}\right)$. – *There exists a constant $C_H = C_H(Y)$ such that for all balls $B_{2r}(x) \subset Y$ and all $t \in \mathbb{R}$*

$$\sup_{(s, y) \in Q^-} u(s, y) \le C_H \cdot \inf_{(s, y) \in Q^+} u(s, y) \tag{3.3}$$

whenever u is a nonnegative local solution of the parabolic equation $\left(L - \frac{\partial}{\partial t}\right) u = 0$ on $Q =]t - 4r^2, t[\times B_{2r}(x)$. Here $Q^- =]t - 3r^2, t - 2r^2[\times B_r(x)$ and $Q^+ =]t - r^2, t[\times B_r(x)$.

In order to be precise, one should replace the "inf" and "sup" in (3.3) by "ess inf" and "ess sup". The following Proposition 3.1, however, states that all functions u under consideration can be chosen to be continuous (more precisely: admit a continuous version). Hence, there is no reason to use this cumbersome notation.

PROPERTY (II*) $\left(\text{Parabolic Harnack inequality for the operators } L_t - \frac{\partial}{\partial t}\right)$. – *For all $\kappa \ge 1$ and all $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ with $0 < \alpha < \beta < \gamma < \delta$ and $0 < \varepsilon < 2$ there exists a constant $C_H^* = C_H^*(Y)$ such that for all balls $B_{2r}(x) \subset Y$ and all $t \in \mathbb{R}$*

$$\sup_{(s, y) \in Q^-} u(s, y) \le C_H^* \cdot \inf_{(s, y) \in Q^+} u(s, y) \tag{3.4}$$

whenever $(L_t)_{t \in R}$ is a uniformly parabolic operator satisfying (3.1) *and u is a nonnegative local solution of the parabolic equation $\left(L_t - \frac{\partial}{\partial t}\right) u = 0$ on $Q =]t - \delta r^2, t[\times B_{2r}(x)$. Here $Q^- =]t - \gamma r^2, t - \beta r^2[\times B_{\varepsilon r}(x)$ and $Q^+ =]t - \alpha r^2, t[\times B_{\varepsilon r}(x)$.*

It is a well known fact that the parabolic Harnack inequality is quite a powerful property which has many important consequences. We will state only some of them.

PROPOSITION 3.1 (Hölder continuity). – *Assume* (II^*). *Then there exist constants* $\alpha \in]0, 1[$ *and* C *such that for all balls* $B_{2r}(x) \subset Y$ *and all* $T \in \mathbb{R}$

$$|u(s, y) - u(t, z)| \leq C \cdot \sup_{Q_2} |u| \cdot \left(\frac{|s-t|^{1/2} + |y-z|}{r} \right)^{\alpha} \tag{3.5}$$

whenever u *is a local solution of the parabolic equation* $\left(L_t - \dfrac{\partial}{\partial t} \right) u = 0$ *on* $Q_2 =]T - 4r^2, T[\times B_{2r}(x)$ *and* (s, y) *and* (t, z) *are points in* $Q_1 =]T - r^2, T[\times B_r(x)$.

Proof. – The proof of Moser [Mo1] carries over without any essential change. □

Of course, the precise assertion of 3.1 is that any local solution of the equation $\left(L_t - \dfrac{\partial}{\partial t} \right) u = 0$ admits a version which satisfies (3.5). This continuous version is uniquely determined and without restriction we will in the sequel always assume all local solutions of that equation are chosen to be continuous.

PROPOSITION 3.2 (Elliptic Harnack inequality). – *Assume (II). Then there exists a constant* C *such that for all balls* $B_{2r}(x) \subset X$

$$\sup_{y \in B_r(x)} u(y) \leq C \cdot \inf_{y \in B_r(x)} u(y) \tag{3.6}$$

whenever u *is a nonnegative local solution of the elliptic equation* $Lu = 0$ *on* $B_{2r}(x)$.

An obvious consequence either of 3.1 or of 3.2 is the following

COROLLARY 3.3 (Hölder continuity). – *Assume (II). Then there exist constants* $\alpha \in]0, 1[$ *and* C *such that for all balls* $B_{2r}(x) \subset Y$

$$|u(y) - u(z)| \leq C \cdot \sup_{B_{2r}(x)} |u| \cdot \left(\frac{|y-z|}{r} \right)^{\alpha} \tag{3.7}$$

whenever u *is a local solution of the elliptic equation* $Lu = 0$ *on* $B_{2r}(x)$ *and* y *and* z *are points in* $B_r(x)$.

A standard consequence of a *global* elliptic Harnack inequality is

COROLLARY 3.4. (Strong Liouville property). – *Assume (II) with* $Y = X$. *Then all nonnegative local solutions of the equation* $Lu = 0$ *on X are constant on X.*

Proof. For any nonnegative local solution u on X put $a = \inf_X u$ and consider $u_1 = u - a$ which is again a nonnegative local solution of $Lu_1 = 0$ on X. Then $\sup_X u_1 \leq C \cdot \inf_X u_1 = C \cdot (\inf_X u - a) = 0$. That is, $u_1 \equiv 0$ on X and thus $u \equiv a$. □

C) Poincaré and parabolic Harnack inequalities

One of the main goals of this paper is the following result on the equivalence of parabolic Harnack inequality and Poincaré inequality together with doubling property.

THEOREM 3.5. – *Under the completeness assumption (Ia) the following are equivalent:*

(i) *The doubling property (Ib) and the Poincaré inequality (Ic) hold true on Y.*

(ii) *The parabolic Harnack inequality (II) for the (time-independent) operator $L - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Y$ holds true.*

(iii) *The parabolic Harnack inequality (II*) holds true for all (time-dependent) operators $L_t - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Y$ which satisfy* (3.1).

The parabolic Harnack constant C_H in (iii) *can be chosen as $C_H(N, C_P)$,* i.e., *only to depend on the doubling constant N and the Poincaré constant C_P. (The constant C_H^* in addition depends on the parabolicity constant and the parameters α, β, γ, δ, ε.) In the converse direction, both constants N and C_P in* (i) *can be chosen to depend only on the parabolic Harnack constant C_H for $L - \frac{\partial}{\partial t}$ $\left(\text{e.g. } N = 4\frac{\ln C_H}{\ln 2} \text{ and } C_P = C_H^2 \cdot 2^N = C_H^6\right)$.*

Proof. – For the implication (i) $\Rightarrow$ (iii) we follow the second proof of Moser [Mo2]. In [St2], Theorem 2.1, we already have deduced the relevant sub- and supersolution estimates which follow by Moser's iteration from the Sobolev inequality of Theorem 2.6 (more precisely, from inequality (2.11)). These estimates are the analogue to Lemma 1 in [Mo2].

The analogue to Lemma 2 in [Mo2] follows from the weighted Poincaré inequality of Corollary 2.5 and from the gradient estimate for the cut-off functions in our Lemma 1.1. In order to see this, one just has to go through pp. 121-123 of [Mo1] and there one has to replace the weight function $p = \psi^2$ by the function ϕ^2 from our Corollary 2.5 (which plays the role of Lemma 3 in [Mo1]). Due to the abstract Lemma 3 in [Mo2], these properties already prove the Harnack inequality.

The implication (iii) $\Rightarrow$ (ii) is trivial. For the implication (ii) $\Rightarrow$ (i), we follow the proof by Saloff-Coste [Sa2] which carries over line by line. The estimates for N and for C_P follow from analysing the dependence in Saloff-Coste's proof. □

COROLLARY 3.6. – *Let $(\tilde{\mathcal{E}}, \mathcal{F})$ be a strongly local and symmetric Dirichlet form on a Hilbert space $L^2(X, \tilde{m})$ which is quasi-isometric to the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ in the sense that $1/\kappa \cdot \mathcal{E} \leq \tilde{\mathcal{E}} \leq \kappa \cdot \mathcal{E}$ (as quadratic forms on $\mathcal{F}$) and $1/\kappa \cdot m \leq \tilde{m} \leq \kappa \cdot m$ and let $(\mathcal{E}, \mathcal{F})$ satisfy (Ia). Then the following are equivalent:*

(i) *The parabolic Harnack inequality (II) holds true for $L - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Y$.*

(ii) *The parabolic Harnack inequality (II) holds true for $\tilde{L} - \frac{\partial}{\partial t}$ on $\mathbb{R} \times Y$.*

Proof. – First of all, note that the quasi-isometry implies that also $(\tilde{\mathcal{E}}, \mathcal{F})$ is strongly regular. The claim follows immediately from the above Theorem as soon as one has shown that the completeness property, the doubling property and the Poincaré inequality are preserved (with new constants depending on κ) under quasi-isometric changes. To this end, note that the assumptions on $\tilde{\mathcal{E}}$ and $\tilde{m}$ imply $1/\kappa \cdot \rho \leq \tilde{\rho} \leq \kappa \cdot \rho$ and thus obviously the completeness property (Ia) is preserved. Let us now assume that the doubling property (Ib) holds for the form $\mathcal{E}$ on $L^2(X, m)$. Then it also holds for $\tilde{\mathcal{E}}$ on $L^2(X, \tilde{m})$ since

$$\tilde{m}(\tilde{B}_{2r}(x)) \leq \kappa \cdot m(B_{2\kappa r}(x)) \leq \kappa \cdot (4\kappa^2)^N \cdot m(B_{r/\kappa}(x)) \leq \kappa^2 \cdot (4\kappa^2)^N \cdot \tilde{m}(\tilde{B}_r(x)).$$

Finally assume that the weak Poincaré inequality (Ic) holds for the form $\mathcal{E}$ on $L^2(X, m)$. Then

$$\begin{aligned}\int_{\tilde{B}_r(x)} |u - \tilde{u}_{x,r}|^2 \, d\tilde{m} &= \inf_{\alpha \in \mathbb{R}} \int_{\tilde{B}_r(x)} |u - \alpha|^2 \, d\tilde{m} \\ &\leq \kappa \cdot \inf_{\alpha \in \mathbb{R}} \int_{B_{\kappa r}(x)} |u - \alpha|^2 \, dm = \kappa \cdot \int_{B_{\kappa r}(x)} |u - u_{x,\kappa r}|^2 \, dm \\ &\leq \kappa^3 \cdot C_P \cdot r^2 \int_{B_{2\kappa r}(x)} d\Gamma(u, u) \leq \kappa^4 \cdot C_P \cdot r^2 \int_{\tilde{B}_{2\kappa^2 r}(x)} d\tilde{\Gamma}(u, u).\end{aligned}$$

By means of a straightforward covering argument (derived from properties (Ia) and (Ib)) this implies the weak Poincaré inequality (Ic) for the form $\tilde{\mathcal{E}}$ on $L^2(X, \tilde{m})$. □

Remarks. – (i) Note that the parabolic Harnack constant on Y only depends on the doubling constant and the Poincaré constant *on the same set* Y and vice versa.

(ii) For Laplace-Beltrami operators on complete Riemannian manifolds the implication (Ib) & (Ic) ⇒ (II) ⇒ (Ib) were obtained independently by A. A. Grigor'yan [Gr] and L. Saloff-Coste [Sa2]. Saloff-Coste also derived the implication (II) ⇒ (Ic), based on an idea of S. Kusuoka and D. W. Stroock [KS].

(iii) The elliptic Harnack inequality for local Dirichlet operators was already established by Biroli and Mosco [BM1, 2]. In addition to the set of assumptions (Ia), (Ib) and (Ic) they required as a further assumption the validity of a Sobolev inequality similar to (2.10) (which according to our Theorem 2.6 already follows from the other assumptions) but actually even slightly stronger.

(iv) The elliptic Harnack inequality *does not* imply the parabolic one. Even more, it does not imply the doubling property. This observation is due to A. A. Grigor'yan [Gr]. Namely, it is easy to construct complete two-dimensional smooth Riemannian manifolds X with the following properties

- $m(B_r(x_0)) \leq C \cdot r^2$ for $r \to \infty$;
- $m(B_2(x_n)) \geq n \cdot m(B_1(x_n))$ for a sequence $(x_n)_n$ of points in X.

For any manifold as above satisfying the first one of these properties, the elliptic Harnack inequality (3.6) holds true according to a result of Cheng and Yau [CY], Prop. 6. But the second one of these conditions obviously contradicts the doubling property (Ib).

4. Gaussian estimates for the heat kernel

We say that property (I) is satisfied on a set $Y \subset X$ if properties (Ia), (Ib) and (Ic) are simultaneously satisfied on Y. According to Theorem 3.5, property (I) implies the parabolic

Harnack inequality (II). In the sequel we will assume property (I) on an open subset $Y \subset X$ in order to get pointwise estimates on Y for the fundamental solution $p(t, y, s, x)$ of the parabolic operator $L_t - \frac{\partial}{\partial_t}$ on $\mathbb{R} \times X$. We emphasize that p is the fundamental solution on the whole space X whereas our assumptions are only stated on the subspace Y.

A) The upper bound

Using the parabolic Harnack inequality (II*) one easily derives pointwise estimates on $\mathbb{R} \times Y$ for the fundamental solution $p(t, y, s, x)$ of the parabolic operator $L_t - \frac{\partial}{\partial t}$ on $\mathbb{R} \times X$. Here $L_t - \frac{\partial}{\partial t}$ is a time-dependent parabolic operator as introduced in section 3.A). It is always assumed to be uniformly parabolic w.r.t. the elliptic operator L in the sense of (3.1).

On $Y \times Y$ the fundamental solution $p(t, ., s, .)$ is defined as the jointly continuous density of the transition operator T_t^s. That is,

$$T_t^s f(y) = \int_X p(t, y, s, x) f(x) m(dx)$$

for every $f \in L^2(X, m)$, $f = 0$ on $X \backslash Y$, and every $y \in Y$; *see* [St2], sect. 2.4.

THEOREM 4.1. – *Assume (I) on the open set $Y \subset X$. Then there exists a constant C depending only on $N = N(Y)$ and $C_P = C_P(Y)$ such that the following estimate holds true for all points (t_1, y_1) and $(t_2, y_2) \in \mathbb{R} \times Y$ with $t_1 < t_2$.*

$$\begin{aligned} (4.1) \quad p(t_2, y_2, t_1, y_1) &\le C \cdot m^{-1/2}(B_{\sqrt{t}}(y_1)) \cdot m^{-1/2}(B_{\sqrt{t}}(y_2)) \\ &\cdot \exp\left(-\frac{\rho^2(y_1, y_2)}{4\kappa(t_2 - t_1)}\right) \cdot \left(1 + \frac{\rho(y_1, y_2)}{\kappa(t_2 - t_2)}\right)^{2 \cdot N/2} \\ &\cdot \exp(-\lambda \cdot (t_2 - t_1)/\kappa) \cdot (1 + \lambda \cdot (t_2 - t_1)/\kappa)^{1+N/2}. \end{aligned}$$

Here $t = \inf\{t_2 - t_1, R^2\}$ with $R = \inf\{\rho(y_1, X\backslash Y), \rho(y_2, X\backslash Y)\}$ (being $+\infty$ if $X = Y$). Furthermore, $\lambda = \inf\left\{\frac{\mathcal{E}(u, u)}{\|u\|^2} : u \in \mathcal{F}, u \neq 0\right\} \ge 0$ denotes the bottom of the spectrum of the selfadjoint operator $-L$ on $L^2(X, m)$.

Proof. – [St2], Theorem 2.4.

Remarks. – (i) The number λ in the estimate (4.1) can always be replaced by 0. That is, the last two terms on the RHS of (4.1) can always be dropped.

Let us mention that the bottom of the spectrum of $-L$ is zero if X is complete and if the volume of balls grows subexponentially ([St1], Theorem 5). For instance, the latter is satisfied if the doubling property (Ib) holds true globally on X (which even implies that the volume grows at most polynomially).

(ii) The polynomial correction term $\left(1+\frac{\rho(x,y)^2}{t}\right)^{N/2}$ in the above estimate (4.1) can of course be absorbed by the Gaussian term $\exp\left(-\frac{\rho^2(x,y)}{4t}\right)$ if we replace the number 4 by some larger one. That is, *for every* $\varepsilon > 0$ *there exists a constant* C *such that*

$$\text{(4.2)}\quad p(t_2, y_2, t_1, y_1) \le C\cdot m^{-1/2}(B_{\sqrt{t}}(y_1))\cdot m^{-1/2}(B_{\sqrt{t}}(y_2))\cdot \exp\left(-\frac{\rho^2(y_1,y_2)}{(4+\varepsilon)\,\kappa\,(t_2-t_1)}\right)$$

for all $y_1, y_2 \in Y$ *and* $t_1 < t_2$ *and with* t *as above.*

(iii) In addition to the assumptions of Theorem 4.1 assume that there exists a geodesic γ joining y_1 and y_2 and that the doubling property (Ib) holds true on the neighborhood $B_R(\gamma) = \{x\in X : \rho(x,\gamma) < R\}$ of γ. Then the usual chaining argument yields

$$\text{(4.3)}\quad m(B_{\sqrt{t}}(y_1)) \le C\cdot m(B_{\sqrt{t}}(y_2))\cdot \exp\left(C\cdot\frac{\rho(y_1,y_2)}{\sqrt{t}}\right).$$

Hence, *for every* $\varepsilon > 0$ *there exists a constant* C *such that*

$$\text{(4.4)}\quad p(t_2, y_2, t_1, y_1) \le C\cdot m^{-1}(B_{\sqrt{t}}(y_1))\cdot \exp\left(-\frac{\rho^2(y_1,y_2)}{(4+\varepsilon)\,\kappa\,(t_2-t_1)}\right)$$

for all $y_1, y_2 \in Y$ *and* $t_1 < t_2$ *and with* t *as above.*

The Gaussian estimates for the fundamental solutions have a particular nice form if the assumption (I) holds true globally on X and if the parabolic operator is time-independent (that is, $L_t = \tilde{L}$ for all $t\in\mathbb{R}$). In this case, without restriction we may assume that $\tilde{L} = L$, that is, $L_t \equiv L$ for all $t\in\mathbb{R}$ (and $\kappa = 1$). The fundamental solution $p(t, y, s, x)$ satisfies $p(t, y, s, x) = p(t-s, y, 0, x)$ and instead of the latter we simply write $p(t-s, y, x)$ and call it heat kernel.

COROLLARY 4.2. – *If property (I) holds true globally on* X *and if* $L_t \equiv L$*, then the following estimate holds true uniformly for all points* $x, y \in X$ *and all* $t > 0$*:*

$$\text{(4.5)}\quad p(t, x, y) \le C\cdot m^{-1/2}(B_{\sqrt{t}}(x))\cdot m^{-1/2}(B_{\sqrt{t}}(y)) \cdot\exp\left(-\frac{\rho^2(x,y)}{4t}\right)\cdot\left(1+\frac{\rho(x,y)^2}{t}\right)^{N/2}.$$

Estimates similar to (4.5) also hold for the time derivatives $\left(\frac{\partial}{\partial t}\right)^k p(t, x, y)$ of the heat kernel. *See* [St2] for details.

B) The lower bound on the diagonal

In the sequel, we are going to deduce lower estimates on Y for the fundamental solution $p(t_2, y_2, t_1, y_1)$ of the parabolic operator $L_t - \frac{\partial}{\partial t}$ on $\mathbb{R} \times X$. We do *not* require that X is complete. Actually, we will impose no conditions on the operators L_t outside of Y. The first and most important step is to get a lower bound for the fundamental solution $p(t_2, y_2, t_1, y_1)$ on the diagonal $\{y_1 = y_2\}$.

THEOREM 4.3. – *Assume (I) on Y. Then there exists a constant (depending only on N and C_P) such that*

$$p(t_2, x, t_1, x) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(x)) \tag{4.6}$$

for all $x \in Y$ and all $t_1, t_2 \in \mathbb{R}$ with $0 < t_2 - t_1 < \rho^2(x, X \backslash Y)$.

The most difficulties in proving this Theorem arise from the fact that we try to avoid any kind of quantitative assumptions (like (Ib), (Ic) or (II)) on $X \backslash Y$.

The idea is to fix a "large" ball $B_{4R}(x) \subset Y$ and to study the fundamental solution p' of the parabolic operator $L_t - \frac{\partial}{\partial t}$ on $X' = \overline{B}_{2R}(x)$ with "reflecting" or "Neumann" boundary conditions on $\partial B_{2R}(x)$. In a second step, the estimates for p' on $B_R(x)$ will be used to derive lower estimates on $B_{R/2}(x)$ for the fundamental solution p'' of the parabolic operator $L_t - \frac{\partial}{\partial t}$ on $X'' = B_R(x)$ with "absorbing" or "Dirichlet" boundary conditions on $\partial B_R(x)$. The final step consists of the simple observation that the original fundamental solution p on X always satisfies $p \geq p''$.

To make these ideas concrete, fix $R > 0$ and $x \in Y$ with $B_{4R}(x) \subset Y$ and put $X' := \overline{B}_{2R}(x)$. Let

$$\mathcal{E}'(u, v) = \int_{X'} d\Gamma(u, v)$$

for $u, v \in \mathcal{F}$. Then $\mathcal{E}'$ is a closable bilinear form on $L^2(X', m)$. Its closure will be denoted by $(\mathcal{E}', \mathcal{F}')$. This form is again regular, symmetric and strongly local. Moreover, it is also strongly regular (since the "new" intrinsic metric is just the "old" one). Analogously, define the forms $(\mathcal{E}'_t, \mathcal{F}')$. Note that the constant function $u \equiv 1$ lies in $L^2(X', m)$ and it is a *global* solution of the equation $\left(L'_t - \frac{\partial}{\partial_t}\right) u = 0$ on $\mathbb{R} \times X'$. Hence, the associated transition operators $(T'^{s}_t)_{s<t}$ are conservative.

LEMMA 4.4. – *Under the above assumptions, there exists a constant (depending only on N and C_P) such that*

$$T'^{t_1}_{t_2} 1_{\overline{B}_{2R}(x) \backslash B_r(y_1)}(y_1) \leq C \cdot \exp\left(-\frac{r^2}{10\,t}\right), \tag{4.7}$$

for all $y_1 \in B_R(x)$ and all $t_1, t_2, r \in \mathbb{R}$ with $0 < t := t_2 - t_1 \leq r^2 \leq R^2/4$.

Proof. – We combine the integrated estimate for p' from [St2] (Corollary 1.11) with an application of the parabolic Harnack inequality on the cube $]t_2, t_2 + t[\times B_{\sqrt{t}}(y_1)$ in order to obtain

$$\begin{aligned}
T'^{t_1}_{t_2} 1_{\overline{B}_{2R}(x)\setminus B_r(y_1)} &\leq \frac{C_H^*}{t \cdot m(B_{\sqrt{t}}(y_1))} \\
&\cdot \sum_{k=1}^{\infty} \int_{B_{\sqrt{t}}(y_1)} \int_{t_2}^{t_2+t} T'^{t_1}_s 1_{\overline{B}_{2R}(x)\cap(B_{(k+1)r}(y_1)\setminus B_{kr}(y_1))}(y)\, ds\, m(dy) \\
&\leq C_H^* \cdot \sum_{k=1}^{\infty} m^{+1/2}(B_{2R}(x) \cap B_{(k+1)r}(y_1)) \cdot m^{-1/2}(B_{\sqrt{t}}(y_1)) \cdot \exp\left(-\frac{(kr)^2}{8t}\right) \\
&\leq C_H^* \cdot \sum_{k=1}^{\infty} \left(\frac{2(k+1)r}{\sqrt{t}}\right)^{N/2} \cdot \exp\left(-\frac{(kr)^2}{8t}\right) \leq C \cdot \exp\left(-\frac{r^2}{10t}\right). \quad \square
\end{aligned}$$

Lemma 4.4 has two important Corollaries. The first one is

LEMMA 4.5. – *Under the above assumptions, there exists a constant (depending only on N and C_P) such that*

$$p'(t_2, y_2, t_1, y_1) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(y_1)) \tag{4.8}$$

for all $t_1, t_2 \in \mathbb{R}$ and $y_1, y_2 \in B_R(x)$ with $\rho^2(y_1, y_2) \leq t := t_2 - t_1 \leq R^2$.

Proof. – Let k be the smallest integer ≥ 2 and $\geq 10 \ln(2C)$ with C being the constant from Lemma 4.4. Then

$$T'^{s_1}_{s_2} 1_{\overline{B}_{2R}(x)\setminus B_r(y_1)}(y_1) \leq \frac{1}{2}$$

whenever $0 < s_2 - s_1 < r^2/k$. Since the transitions operators T'^s_t are conservative on $\overline{B}_{2R}(x)$, this is equivalent to

$$T'^{s_1}_{s_2} 1_{B_r(y_1)}(y_1) = \int_{B_r(y_1)} p'(s_2, y_1, s_1, y)\, m(dy) \geq \frac{1}{2} \tag{4.9}$$

whenever $0 < s_2 - s_1 < r^2/k$.

For given $t_1 < t_2$ and $i = 1, \ldots, k+1$ put $s_i := t_1 + (i-1)\dfrac{t_2 - t_1}{k}$. Then $s_1 = t_1$ and $s_{k+1} = t_2$. The adjoint version of the parabolic Harnack inequality (II*) will be applied to the function $(s, y) \mapsto p'(s_k, y_2, s, y)$ which is a solution of the adjoint equation $\left(L_t + \dfrac{\partial}{\partial t}\right) u = 0$ on $]-\infty, s_k[\times X$. This implies

$$
\begin{aligned}
p'(s_{k+1}, y_2, s_1, y_1) &\geq \frac{1}{C_H^*} \cdot p'(s_{k+1}, y_2, s_2, y_1) \\
&\geq \ldots \geq \left(\frac{1}{C_H^*}\right)^{k-2} \cdot p'(s_{k+1}, y_2, s_{k_1}, y_1) \\
&\geq \left(\frac{1}{C_H^*}\right)^{k-1} \cdot m^{-1}(B_{\sqrt{t}}(y_2)) \cdot \int_{B_{\sqrt{t}}(y_2)} p'(s_{k+1}, y_2, s_k, y)\, m(dy) \\
&\geq \left(\frac{1}{C_H^*}\right)^{k-1} \cdot m^{-1}(B_{\sqrt{t}}(y_2)) \cdot \frac{1}{2},
\end{aligned}
$$

for all $y_2 \in B_{\sqrt{t}}(y_1)$ where $t := t_2 - t_1 < R^2$. □

In order to formulate the second corollary to Lemma 4.4, let

$$(\mathbb{P}^{t_0, y_0}, (t_0 - t, X_{t_0 - t}))_{(t_0, y_0) \in E, t \geq 0}$$

be the diffusion process on $E := \mathbb{R} \times \overline{B}_{2R}(x)$ which is associated with the family of Dirichlet forms $\{\mathcal{E}'_t\}_{t \in R}$ in such a way that

$$\mathbb{E}^{t, y}[f(X_s);\ s > \xi] = T_t'^s f(x)$$

for every $s, t \in \mathbb{R}$ with $s < t$, for every quasi-continuous $f \in L^2(\overline{B}_{2R}(x), m)$ and for q.e. $y \in \overline{B}_{2R}(x)$, *see* Y. Oshima [Os]. Note that under the above assumptions also the form $(\mathcal{E}', \mathcal{F}')$ has the property (I), at least on the open ball $B_{2R}(x)$ (where it essentially coincides with the original form). Therefore, the process can be defined uniquely for every starting point in $B_{2R}(x)$ (and not only for q.e.).

LEMMA 4.6. – *Under the above assumptions, there exists a constant (depending only on N and C_P) such that*

$$\mathbb{P}^{t_2, y_2}\left(\sup_{t_1 \leq s \leq t_2} \rho(X_s, y_2) \geq r\right) \leq C \cdot \exp\left(-\frac{r^2}{10(t_2 - t_1)}\right) \tag{4.10}$$

for all $y_1 \in B_R(x)$ and all $t_1 < t_2$.

Proof. – The arguments from the proof of Lemma 3 in [SaS] apply without essential changes. □

Now we are in a position to derive the lower estimate for the fundamental solution p'' for the parabolic operator $L_t - \dfrac{\partial}{\partial t}$ on $X'' := B_R(x)$ with absorbing boundary conditions on $\partial B_R(x)$. To be precise, let $(\mathcal{E}'', \mathcal{F}'')$ be the Dirichlet form on $L^2(B_R(x), m)$ obtained either from $(\mathcal{E}, \mathcal{F})$ or (equivalently!) from $(\mathcal{E}', \mathcal{F}')$ by defining $\mathcal{F}'' := \{u \in \mathcal{F} : \tilde{u} = 0$ q.e. on $X \backslash B_R(x)\} = \{u \in \mathcal{F}' : \tilde{u} = 0$ q.e. on $\overline{B}_{2R}(x) \backslash B_R(x)\}$ (where $\tilde{u}$ denotes a quasi-continuous version of u) and $\mathcal{E}''(u, v) := \mathcal{E}(u, u) = \mathcal{E}'(u, u)$ for all $u, v \in \mathcal{F}''$. Analogously, define the forms $(\mathcal{E}''_t, \mathcal{F}'')$ for $t \in \mathbb{R}$. All these Dirichlet forms are again strongly regular, symmetric and strongly local. Denote the transition operators associated with the family of Dirichlet forms $\{(\mathcal{E}''_t, \mathcal{F}'')\}_{t \in R}$ by $T_t''^s$, $s < t$, and the associated fundamental solution by $p''(t, y, s, x)$. Note that its existence is guaranteed since the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ was assumed to satisfy property (I) on $Y \supset B_{4R}(x)$ and thus the new Dirichlet form $(\mathcal{E}'', \mathcal{F}'')$ satisfies property (I) on each set $B_r(x)$ with $r < R$.

LEMMA 4.7. – *Under the above assumptions, there exists a constant (depending only on N and C_P) such that*

$$p''(t_2, x, t_1, x) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(y_1)) \tag{4.11}$$

for all $t_1, t_2 \in \mathbb{R}$ with $0 < t := t_2 - t_1 \leq R^2$.

Proof. – From the strong Markov property for the diffusion process $(\mathbb{P}^{t,x}, (t-s, X_{t-s}))$ one deduces that

$$p''(t_2, x, t_1, x) = p'(t_2, x, t_1, x) - \mathbb{E}^{t_2, x}[p'(\sigma, X_\sigma, t_1, x); \sigma > t_1] \tag{4.12}$$

where $\sigma = \sup\{s \leq t_2 : X_s \in \overline{B}_{2R}(x) \backslash B_R(x)\}$ (with $\sup \emptyset := -\infty$). The first term on the RHS can be estimated from below by Lemma 4.5 and the second term from above by Lemma 4.6 together with Theorem 4.1. Note that for $t_1 \leq \sigma \leq t_2$ and $X_\sigma \in \partial B_R(x)$ the estimate from Theorem 4.1 implies:

$$\begin{aligned} p'(\sigma, X_\sigma, t_1, x) &\leq C \cdot m^{-1}(B_{\sqrt{t_1 - \sigma}}(x)) \cdot \exp\left(-\frac{R^2}{5(t_1 - \sigma)}\right) \\ &\leq C' \cdot \left(\frac{t}{t_1 - \sigma}\right)^{N/2} \cdot m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-\frac{R^2}{5(t_1 - \sigma)}\right) \\ &\leq C'' \cdot \left(\frac{t}{R^2}\right)^{N/2} \cdot m^{-1}(B_{\sqrt{t}}(x)). \end{aligned}$$

Hence, we get

$$\begin{aligned} p''(t_2, x, t_1, x) &\geq \frac{1}{C_1} \cdot m^{-1}(B_{\sqrt{t}}(x)) \\ &\quad - \left[C_2 \cdot \exp\left(-\frac{R^2}{10\,t}\right)\right] \cdot \left[C_3 \cdot \left(\frac{t}{R^2}\right)^{N/2} \cdot m^{-1}(B_{\sqrt{t}}(x))\right] \\ &\geq \frac{1}{2\,C_1} \cdot m^{-1}(B_{\sqrt{t}}(x)) \end{aligned}$$

provided $\frac{t}{R^2}$ is sufficiently small, *i.e.* $\leq 1/k$ with a constant $k \in \mathbb{N}$ depending only on N and C_P.

This yields the claim for all $t_1, t_2 \in \mathbb{R}$ with $t_2 - t_1 \leq R^2/k$. The general case follows by applying the parabolic Harnack inequality (II*) at times $s_i := t_1 + (i-1)(t_2 - t_1)/k$, $i = 1, \ldots, k+1$ in order to obtain

$$
\begin{aligned}
p''(s_{k+1}, x, s_1, x) &\geq \frac{1}{C_H^*} \cdot p''(s_k, x, s_1, x) \\
&\geq \dots \geq \left(\frac{1}{C_H^*}\right)^{k-1} \cdot p''(s_2, x, s_1, x) \\
&\geq \left(\frac{1}{C_H^*}\right)^{k-1} \cdot \frac{1}{2C} \cdot m^{-1}(B_{\sqrt{s_2 - s_1}}(x)) \\
&\geq \left(\frac{1}{C_H^*}\right)^{k-1} \cdot \frac{1}{2C} \cdot m^{-1}(B_{\sqrt{t}}(x)). \quad \square
\end{aligned}
$$

The proof of Theorem 4.3 is now complete since

$$p(t_2, y_2, t_1, y_1) \geq p''(t_2, y_2, t_1, y_1)$$

for all $y_1, y_2 \in X'' = B_R(x)$ and all $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. $\square$

C) The lower bound off the diagonal

From the on-diagonal estimate (4.6) one easily deduces an off-diagonal estimate by the usual chaining argument.

THEOREM 4.8. – *Assume that (I) is satisfied on a set $Y \subset X$. Then there exists a constant C (depending only on N and C_P) such that*

$$
\begin{aligned}
(4.13) \qquad p(t_2, y_2, t_1, y_1) \geq & \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(y_1)) \\
& \cdot \exp\left(-C \frac{\rho^2(y_1, y_2)}{t_2 - t_1}\right) \cdot \exp\left(-\frac{C}{R^2}(t_2 - t_1)\right)
\end{aligned}
$$

for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and all points $y_1, y_2 \in Y$ which are joined in Y by a curve γ of length $\rho(y_1, y_2)$. Here $t = \inf\{t_2 - t_1, R^2\}$ with $R = \inf_{0 \leq s \leq 1} \rho(\gamma(s), X \backslash Y)$ (being $+\infty$ if $X = Y$).

Proof. – Let t be as above, let k be the smallest integer $\geq \frac{\rho^2(y_1, y_2)}{t}$ and let l be the smallest integer $\geq \frac{t_2 - t_1}{R^2}$. Put $(s_i, x_i) = (t_1 + (i-1) \cdot R^2, y_1)$ for $i = 1, \dots, l-1$ and $(s_{l+i}, x_{l+i}) = (t_2 - (k-i) \cdot t/k, \gamma(i/k))$ for $i = 0, \dots, k$ where we assume that $\gamma : [0, 1] \to Y$ is parametrized proportional to arclength and such that $\gamma(0) = y_1$ and $\gamma(1) = y_2$. Then $(s_1, x_1) = (t_1, y_1)$ and $(s_{l+k}, x_{l+k}) = (t_2, y_2)$. Moreover,

$$\rho^2(x_{i+1}, x_i) \leq s_{i+1} - s_i \leq R^2$$

for all $i = 1, ..., l+k$. Hence, the parabolic Harnack inequality (II*) yields

$$\begin{aligned} p(s_{l+k}, x_{l+k}, s_1, x_1) &\geq \frac{1}{C_H^*} \cdot p(s_{l+k-1}, x_{l+k-1}, s_1, x_1) \\ &\geq \dots \geq \left(\frac{1}{C_H^*}\right)^{l+k-2} \cdot p(s_2, x_2, s_1, x_1) \\ &\geq \frac{1}{C} \cdot \left(\frac{1}{C_H^*}\right)^{l+k-2} \cdot m^{-1}(B_{\sqrt{s_2-s_1}}(x_1)) \\ &\geq \frac{1}{C} \cdot \left(\frac{1}{C_H^*}\right)^{l+k-2} \cdot m^{-1}(B_{\sqrt{t}}(x_1)) \\ &\geq \frac{1}{C} \left(\frac{1}{C_H^*}\right)^{\frac{t_2-t_1}{R^2} + \frac{\rho^2(y_1, y_2)}{t}} \cdot m^{-1}(B_{\sqrt{t}}(x_1)). \quad \square \end{aligned}$$

Remarks. – (i) With the same assumptions and notations as in the above Theorem we have the estimates

$$\begin{aligned} p(t_2, y_2, t_1, y_1) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(y_2)) \\ \cdot \exp\left(-C \frac{\rho^2(y_1, y_2)}{t_2 - t_1}\right) \cdot \exp\left(-\frac{C}{R^2}(t_2 - t_1)\right) \end{aligned}$$

and

$$\begin{aligned} p(t_2, y_2, t_1, y_1) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(y_1)) \cdot m^{-1}(B_{\sqrt{t}}(y_2)) \\ \cdot \exp\left(-C \frac{\rho^2(y_1, y_2)}{t_2 - t_1}\right) \cdot \exp\left(-\frac{C}{R^2}(t_2 - t_1)\right) \end{aligned}$$

This follows immediately from Theorem 4.10 and the volume estimate (4.3).

(ii) Note that the last term on the RHS of (4.13) vanishes if $Y = X$. In the general case, the last term on the RHS of (4.13) should be compared with the last term on the RHS of (4.1). One always has

$$\frac{2^{N+2}}{R^2} \geq \lambda^{\mathrm{Dir}}(B_R(y_1)) \geq \lambda^{\mathrm{Dir}}(Y) \geq \lambda^{\mathrm{Dir}}(X) \geq 0$$

due to (2.4) and the monotonicity of the Dirichlet spectral bound.

COROLLARY 4.10. – *If (I) is satisfied globally and if $L_t \equiv L$, then there exists a constant C (depending only on N and C_P) such that*

$$p(t, x, y) \geq \frac{1}{C} \cdot m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-C \cdot \frac{\rho^2(x, y)}{t}\right) \tag{4.14}$$

for all $t > 0$ and all $x, y \in X$.

D) Application to Green function estimates

Throughout this section, let $L_t \equiv L$. One easily deduces from heat kernel estimates like (4.5) and (4.14) estimates for the *Green function* $g(x, y) = \int_0^\infty p(t, x, y)\, dt$.

COROLLARY 4.11. – *If (I) is satisfied globally, then there exists a constant C such that*

$$C^{-1} \int_{\rho(x,y)}^\infty \frac{r\, dr}{m(B_r(x))} \leq g(x, y) \leq C \int_{\rho(x,y)}^\infty \frac{r\, dr}{m(B_r(x))}$$

for all $x, y \in X$ *where C can be chosen to depend only on N and* C_P.

Proof. – The lower bound follows immediately from (4.14). Namely:

$$\begin{aligned} g(x, y) &\geq \frac{1}{C} \cdot \int_0^\infty m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-C \cdot \frac{\rho^2(x, y)}{t}\right) dt \\ &\geq \frac{1}{C'} \cdot \int_{\rho^2(x,y)}^\infty m^{-1}(B_{\sqrt{t}}(x))\, dt = \frac{2}{C'} \cdot \int_{\rho(x,y)}^\infty m^{-1}(B_r(x))\, r\, dr. \end{aligned}$$

The upper bound is slightly more complicated. From (4.5) it follows that:

$$\begin{aligned} g(x, y) &\leq C \cdot \int_0^\infty m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-\frac{\rho^2}{8t}\right) dt \\ &\leq C \cdot \int_0^{\rho^2} m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-\frac{\rho^2}{8t}\right) dt \\ &\quad + C. \int_{\rho^2}^\infty m^{-1}(B_{\sqrt{t}}(x)) \cdot \exp\left(-\frac{\rho^2}{8t}\right) dt \\ &\leq C \cdot m^{-1}(B_\rho(x)) \cdot \int_0^{\rho^2} \left(\frac{2\rho}{\sqrt{t}}\right)^N \exp\left(-\frac{\rho^2}{8t}\right) dt + C \cdot \int_{\rho^2}^\infty m^{-1}(B_{\sqrt{t}}(x))\, dt \\ &\leq C' \cdot m^{-1}(B_\rho(x)) \cdot \rho^2 + C \cdot \int_{\rho^2}^\infty m^{-1}(B_{\sqrt{t}}(x))\, dt \\ &\leq C'' \cdot \int_{\rho^2}^\infty m^{-1}(B_{\sqrt{t}}(x))\, dt \end{aligned}$$

where we used the abbreviation $\rho = \rho(x, y)$ □.

This type of Green function estimate was already obtained in full generality by M. Biroli and U. Mosco [BM1, 2]. It is well-known in more concrete situations like in Riemannian geometry or in the theory of subelliptic operators. From the upper bound (4.4) for the Green function we easily derive a necessary and sufficient criterion for recurrence, *see* [St2]. In Riemannian geometry, this criterion was established by N. Varopoulos [Va].

COROLLARY 4.12. – *Let the Dirichlet form $\mathcal{E}$ be irreducible and let (I) hold true globally on X. Moreover, fix an arbitrary point $x \in X$. Then $\mathcal{E}$ is recurrent if and only if*

$$\int_1^\infty \frac{r\, dr}{m(B_r(x))} = \infty. \tag{4.15}$$

5. Examples

The main examples which we have in mind are:

A) Laplace-Beltrami operators

Let L be the Laplace-Beltrami operator on a smooth Riemannian manifold and m be the Riemannian volume. In this case, ρ is just the Riemannian distance. Properties (Ia), (Ib) and (Ic) are always satisfied locally on X. Property (Ia) is satisfied for all $Y \subset X$ if the manifold is complete. Properties (Ib) and (Ic) are all satisfied uniformly locally on X (*i.e.* for all balls $B_r(x) \subset X$ with constants depending only on r) if the Ricci curvature on X is bounded from below and they are all satisfied globally on X if the Ricci curvature on X is nonnegative, *cf.* [Sa].

B) Operators with weights

Let L be a uniformly elliptic operator with a nonnegative weight ϕ on $\mathbb{R}^N$, *i.e.* $\mathcal{E}(u, v) = \sum_{i,j=1}^{N} \int a_{ij} \frac{\partial}{\partial x_i} u \cdot \frac{\partial}{\partial x_j} v \cdot \phi \, dx$ and $(u, v) = \int uv \, \phi \, dx$ with $(a_{ij})_{i,j}$ being a symmetric and uniformly elliptic matrix on $\mathbb{R}^N$ and ϕ as well as $\phi^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N, dx)$. In this case, ρ is equivalent to the Euclidean distance and Property (Ia) is always satisfied. If the weight ϕ even belongs to the Muckenhaupt class A_2, then Properties (Ib) and (Ic) are both satisfied globally on X, *cf.* [BM2].

C) Subelliptic operators

Let L be a subelliptic operator on $\mathbb{R}^N$, *i.e.* $\mathcal{E}(u, v) = \sum_{i,j=1}^{N} \int a_{ij} \cdot \frac{\partial}{\partial x_i} u \cdot \frac{\partial}{\partial x_j} v \, dx$ and $(u, v) = \int uv \, dx$ with (a_{ij}) being symmetric and elliptic and such that

$$\mathcal{E}(u, u) \geq \delta \cdot \|u\|^2_{H^\varepsilon} - \|u\|^2$$

for some $\delta, \varepsilon > 0$. In this case, ρ is equal to the metric used e.g. by Fefferman/Phong [FeP], Fefferman/Sanchez-Calle [FeS], Jerison [Je], Jerison/Sanchez-Calle [JS], Nagel/Stein/Wainger [NSW]; it can locally be estimated by the Euclidean distance $|\cdot|$ as follows $\frac{1}{C} \cdot |x - y| \leq \rho(x, y) \leq C \cdot |x - y|^\varepsilon$. Properties (Ia), (Ib) and (Ic) are satisfied globally on X, *cf.* [BM2].

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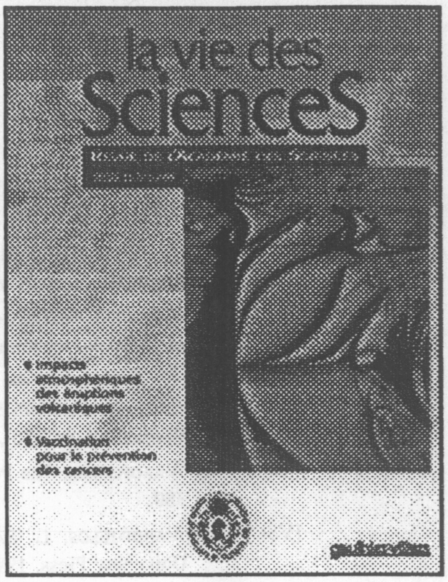

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CONTENTS OF N°3/1995

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