J. reine angew. Math. **456** (1994), 173—196

Journal für die reine und
angewandte Mathematik

Analysis on local Dirichlet spaces

I. Recurrence, conservativeness and L^p-Liouville properties

By *Karl-Theodor Sturm* at Erlangen

---

1. Dirichlet spaces and the intrinsic metric

The basic object for the sequel is a fixed *regular Dirichlet form* $\mathscr{E}$ with domain $\mathscr{D}(\mathscr{E})$ on a real Hilbert space $H = L^2(X, m)$. The underlying topological space X is a locally compact separable Hausdorff space and m is a positive Radon measure with $\mathrm{supp}[m] = X$. The form $\mathscr{E}$ is always assumed to be *strongly local* (i.e. $\mathscr{E}(u, v) = 0$ whenever $u \in \mathscr{D}(\mathscr{E})$ is constant on a neighborhood of the support of $v \in \mathscr{D}(\mathscr{E})$) and to be *irreducible* (i.e. $u \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E})$ is constant on X whenever $\mathscr{E}(u, u) = 0$). In other words, $\mathscr{E}$ has no killing measure and no jumping measure and X cannot be decomposed into (non-trivial) subsets which are invariant for $\mathscr{E}$. For notions concerning Dirichlet forms we recommend the monograph [F] of M. Fukushima whose terminology we mostly follow. For a brief discussion of the notion or irreducibility we refer to the Appendix at the end of this article. Let us mention one crucial consequence of the regularity of $\mathscr{E}$. Each function $u \in \mathscr{D}(\mathscr{E})$ admits a quasi-continuous version $\tilde{u}$ (which is determined pointwise up to exceptional sets). For simplicity, we always write again u for $\tilde{u}$ and make the convention that whenever we use a pointwise version of u (e.g. in expressions like $\int_{\{u>0\}} d\mu$ or $\int u \, d\mu$ with a measure μ charging no exceptional sets) then without restriction this version is always chosen quasi-continuous.

Any Dirichlet form $\mathscr{E}$ as above can be written as

$$\mathscr{E}(u, v) = \int_X d\Gamma(u, v) \tag{1.1}$$

where Γ is a positive semidefinite, symmetric bilinear form on $\mathscr{D}(\mathscr{E})$ with values in the signed Radon measures on X (the so-called energy measure). The nonnegative measure $\Gamma(u, u)$ can be defined by the formula

$$\int_X \phi \, d\Gamma(u, u) = \mathscr{E}(u, \phi u) - \frac{1}{2} \mathscr{E}(u^2, \phi) \tag{1.2}$$

for every $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, m)$ and every $\phi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(X)$. By a straightforward approximation argument, the quadratic form $u \mapsto \Gamma(u, u)$ can be extended to the whole space $L^2_{\text{loc}}(X, m)$ in such a way that $\{u \in L^2_{\text{loc}}(X, m): \Gamma(u, u) \text{ is a Radon measure}\}$ coincides with the set $\mathcal{D}_{\text{loc}}(\mathcal{E})$, being the set of all m-measurable functions u on X for which on every compact set $Y \subset X$ there exists a function $v \in \mathcal{D}(\mathcal{E})$ with $u = v$ m-a.e. on Y. (Similarly, $\mathcal{E}$ can be extended in such a way that $\mathcal{D}(\mathcal{E}) = \{u \in L^2(X, m): \mathcal{E}(u, u) < \infty\}$.) By polarization, we then obtain for arbitrary $u, v \in \mathcal{D}_{\text{loc}}(\mathcal{E})$ a signed Radon measure

$$\Gamma(u, v) = \frac{1}{4}\left(\Gamma(u+v, u+v) - \Gamma(u-v, u-v)\right).$$

The energy measure $\Gamma(u, v)$ does not charge exceptional sets. Moreover, since $\mathcal{E}$ is assumed to be strongly local, the energy measure Γ is local and satisfies the Leibniz rule as well as the chain rule. For these and further properties of the energy measure we refer to M. Fukushima [F], Y. Lejan [Le] and U. Mosco [M] as well as to the Appendix of this paper.

The energy measure Γ defines in an intrinsic way a pseudo metric ϱ on X by

(1.3) $$\varrho(x, y) = \sup\{u(x) - u(y): u \in \mathcal{D}_{\text{loc}}(\mathcal{E}) \cap \mathcal{C}(X),\ d\Gamma(u, u) \leqq dm \text{ on } X\},$$

called intrinsic metric or Carathéodory metric (cf. M. Biroli and U. Mosco [BM1], [BM2] and E. B. Davies [D1]). The condition $d\Gamma(u, u) \leqq dm$ in (1.3) means that the energy measure $\Gamma(u, u)$ is absolutely continuous w.r.t. the reference measure m with Radon-Nikodym derivative $\frac{d}{dm}\Gamma(u, u) \leqq 1$. The density $\frac{d}{dm}\Gamma(u, u)(z)$ should be interpreted as the square of the (length of the) gradient of u at $z \in X$.

In general, ϱ may be degenerate (i.e. $\varrho(x, y) = \infty$ or $\varrho(x, y) = 0$ for some $x \neq y$). Throughout this paper we make the following basic

Assumption (A). *The topology induced by ϱ is equivalent to the original topology on X and all balls $B_r(x) = \{y \in X: \varrho(x, y) < r\}$ are relatively compact in X.*

This assumption in particular implies that ϱ is non-degenerate. It will be discussed in more details in the Appendix where we also give the proof of the following fundamental Lemma.

Lemma 1. *For every $x \in X$ the distance function $\varrho_x: y \mapsto \varrho(x, y)$ on X satisfies $\varrho_x \in \mathcal{D}_{\text{loc}}(\mathcal{E}) \cap \mathcal{C}(X)$ and*

(1.4) $$d\Gamma(\varrho_x, \varrho_x) \leqq dm.$$

In other words, Lemma 1 states that under Assumption (A) the distance function $\varrho_x: y \mapsto \varrho(x, y)$ itself is one of the admissible functions u in the definition (1.3) of $\varrho(x, .)$. The intuitive interpretation of that result is that the distance function ϱ_x is "weakly differentiable" with gradient of length $\leqq 1$ at every point of X. Note that the estimate (1.4) is of course sharp (e.g. in the Euclidean case $X = \mathbb{R}^N$ it states that $|\nabla \varrho_x|^2 \leqq 1$ which is actually an identity on $\mathbb{R}^N \setminus \{x\}$).

According to Lemma 1, the distance function ϱ_x can be used to construct cut-off functions on intrinsic balls $B_r(x)$ of the form

$$(1.5) \qquad \varrho_{x,r}\colon y \mapsto (r - \varrho(x,y))_+ .$$

Obviously, $\varrho_{x,r} \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$ and $d\Gamma(\varrho_{x,r}, \varrho_{x,r}) \leqq dm$ (for all $x \in X$, $r > 0$). The existence of these cut-off functions is the key to prove in our context a series of famous results (by Yau, Cheng/Yau, Karp, Karp/Li, Grigor'yan) which are well-known for the canonical Dirichlet form on complete Riemannian manifolds. Local Dirichlet forms turn out to be an appropriate frame to unify and extend these results.

In section 2 we give a *local* definition for sub- and supersolutions of the equation $Lu = 0$ on X and we prove that under certain *global* integrability conditions these solutions must be constant. In section 3 we give sharp conditions in terms of the volume growth $r \mapsto m(B_r(x))$ for recurrence as well as for conservativeness and for exponential instability.

We emphasize that the scope of applications of the following results is much broader than classical Riemannian geometry. For instance, sub-Riemannian geometry in the sense of R.S. Stichartz [St] is included. The results also apply to uniformly elliptic operators on Riemannian manifolds (cf. [Sa]) as well as to uniformly elliptic operators with weights, to Hörmander type operators and general subelliptic operators on $\mathbb{R}^N$ (cf. [BM2] and references cited therein).

2. Liouville theorems and L^p-uniqueness

The classical Liouville theorem for the canonical Dirichlet form on $\mathbb{R}^N$ states that every nonnegative harmonic function on $\mathbb{R}^N$ must be constant. S.T. Yau [Y1] proved that also every complete Riemannian manifold with nonnegative Ricci curvature has this *strong Liouville property*. However, this curvature condition is not stable under quasi-isometric changes of the Riemannian metric. (Also the strong Liouville property is unstable under quasi-isometric changes, cf. [Ly].) In [Y2] it was proved that (without any further assumption) every complete Riemannian manifold has the L^p-Liouville property for every $p \in \,]1, \infty[$ which means that every harmonic function $u \in L^p$ must be constant.

For various values of $p \in \,]-\infty, \infty[$ we consider the set of nonnegative (global) sub- or supersolutions of the equation $Lu = 0$ on X which satisfy $\int u^p dm < \infty$. We say that the L^p-uniqueness for the subsolutions (resp. supersolutions) is fulfilled if every such function must be constant. Here L denotes the unique negative semidefinite selfadjoint operator on $L^2(X, m)$ associated with $\mathscr{E}$.

Definition. For $\alpha \geqq 0$ a function u on X is called a *subsolution* (resp. *supersolution*) of the equation $(L - \alpha)u = 0$ on an open set $Y \subset X$ if $u \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E})$ and

$$(2.1) \qquad \mathscr{E}(u, \phi) + \alpha \int u \cdot \phi \, dm \leqq 0$$

(or $\mathscr{E}_\alpha(u, \phi) \geqq 0$, resp.) for all nonnegative $\phi \in \mathscr{D}(\mathscr{E}) \cap L^2_{\mathrm{comp}}(Y, m)$ (i.e. $\phi \in \mathscr{D}(\mathscr{E})$ with compact support in Y). Moreover, u is called a *solution* if it is a subsolution and a supersolution. In this case (2.1) holds true with "=" for all $\phi \in \mathscr{D}(\mathscr{E}) \cap L^2_{\mathrm{comp}}(Y, m)$.

Let us make a (perhaps superfluous) comment on the interpretation of statements like "u is nonnegative", "u is bounded" or "u is constant". For functions $u \in L^2_{\text{loc}}(X, m)$ the assertion "u is constant on Y" of course means that there exists a constant $C \in \mathbb{R}$ such that $m\{x \in Y: u(x) \neq C\} = 0$. If Y is open and if $u \in \mathcal{D}_{\text{loc}}(\mathcal{E})$ this is equivalent to the assertion $\operatorname{cap}\{x \in Y: \tilde{u}(x) \neq C\} = 0$ where $\tilde{u}$ is a quasi-continuous version of u.

Solutions (resp. sub-/supersolutions) of the equation $Lu = 0$ on X are also called L-harmonic (or L-sub-/L-superharmonic, resp.) functions on X. A function u in the domain $\mathcal{D}(L)$ of the operator L (or in $\mathcal{D}_{\text{loc}}(L)$) is a subsolution (resp. supersolution) of the equation $Lu = 0$ on X if and only if the inequality $Lu \geqq 0$ (resp. $Lu \leqq 0$) holds true on X. The set of nonnegative subsolutions of the equation $Lu = 0$ is of particular interest since for every solution u of $Lu = 0$ on X the function $|u|$ is a nonnegative subsolution (Lemma 2).

Theorem 1. *Let $u \geqq 0$ be a function on X and for $p \in]-\infty, \infty[$ and some point $x \in X$ let $v(r) = \int_{B_r(x)} u^p dm$ (resp. $v(r) = m(B_r(x))$ if $p = 0$). Assume that*

$$\int_1^\infty \frac{r}{v(r)} dr = \infty. \tag{2.2}$$

a) *If $p < 1$ and if u is a supersolution of the equation $Lu = 0$ on X then u is constant.*

b) *If $p > 1$ and if u is a subsolution of the equation $Lu = 0$ on X then u is constant.*

Note that the condition (2.2) is for instance satisfied if $\int_{B_{r_n}(x)} u^p dm \leqq C \cdot r_n^2$ for a sequence $r_n \to \infty$, in particular, if $\int_X u^p dm < \infty$.

Corollary 1. a) *Let $p \in]-\infty, 1[$. Then every nonnegative supersolution u of the equation $Lu = 0$ on X with $\int u^p dm < \infty$ is constant. In particular, if $m(X) < \infty$ then every nonnegative supersolution u of the equation $Lu = 0$ on X is constant.*

b) *Let $p \in]1, \infty[$. Then every nonnegative subsolution u of the equation $Lu = 0$ on X with $\int u^p dm < \infty$ is constant.*

It is well-known that the assertions of Theorem 1 or Corollary 1 do not hold for $p = 1$ without some additional assumptions (see the following Remark b)). We say that $\mathcal{E}$ (or the heat semigroup $(T_t)_{t>0}$ associated with $\mathcal{E}$) is *conservative* or that X is *stochastically complete* w.r.t. $(T_t)_{t>0}$ if $T_t 1 \equiv 1$ for all $t > 0$. It means that the associated Markov process has infinite life time ($\mathbb{P}^x$-a.s. for q.e. starting point $x \in X$).

Theorem 2. *Assume that $\mathcal{E}$ is conservative. Then every nonnegative supersolution u of the equation $Lu = 0$ on X with $\int u\, dm < \infty$ is constant. Moreover, every nonnegative function $u \in L^1(X, m)$ which is excessive for the semigroup $(T_t)_{t>0}$ or which is defective for the semigroup $(T_t)_{t>0}$ must be constant.*

For the sake of completeness we recall the

Definition. A function $u \in \bigcup_{1 \leqq p \leqq \infty} L^p(X, m)$ is called *excessive* (resp. *defective*) for the semigroup $(T_t)_{t>0}$ if u is nonnegative and if $T_t u \leqq u$ (or $T_t u \geqq u$, resp.) for all $t > 0$. An arbitrary m-measurable function u on X is called excessive for the semigroup $(T_t)_{t>0}$ if u is nonnegative and if $T_t v \leqq u$ for all $t > 0$ and all $v \in \bigcup_{1 \leqq p \leqq \infty} L^p(X, m)$ *with* $v \leqq u$.

Remarks. a) Let us consider our results for the canonical Dirichlet form on a complete Riemannian manifold. In this situation, Corollary 1 was proved by S.T. Yau [Y2]. However, he considered only the cases $p \in]0, 1[$ and $p \in]1, \infty[$ (and he assumed in addition that u is smooth). He also pointed out that instead of assuming $\int_X u^p dm < \infty$ it suffices to assume $\liminf_{r \to \infty} \frac{1}{r} \int_{B_r(x)} u^p dm = 0$. The case $p = 0$ was considered by S.Y. Cheng and S.T. Yau [CY]. They proved that $\liminf_{r \to \infty} \frac{1}{r^2} m(B_r(x)) = 0$ implies that the manifold X is *parabolic*, that is, all nonnegative superharmonic functions (i.e. supersolutions of the Laplace-Beltrami equation $Lu = 0$) on X are constant. The results of [Y2] and [CY] have been improved by L. Karp [K1] who derived essentially the criteria from Theorem 1 for the canonical Dirichlet form on a complete Riemannian manifold (restricting himself to the cases $p > 1$ and $p = 0$).

b) Examples of complete Riemannian manifolds which admit nonconstant *harmonic* functions $u \in L^1$ were constructed by O.S. Chung (unpublished) and P. Li and R. Schoen [LS]. In the latter example the manifold is even stochastically complete.

Conditions which ensure the triviality of the set of nonnegative *subharmonic* functions $u \in L^1$ have been given by P. Li [Li]. However, these conditions are in terms of the Ricci curvature and unstable under quasi-isometric changes. A.A. Grigor'yan [G2] proved that on a complete Riemannian manifold every nonnegative *superharmonic* function $u \in L^1$ is constant provided the manifold is stochastically complete.

A sharp condition on the volume growth which ensures conservativeness will be given in the next section (Theorem 4). Also in the next section the case $p = 0$ ("recurrence") which is of particular importance will be treated again (Theorem 3).

c) The condition (2.2) in Theorem 1 is sharp in the following sense: given a smooth function v (with $v, v' \geqq 0$) which does not satisfy (2.2) then there exists a complete Riemannian manifold X, a point $x \in X$ and a nonconstant (bounded, nonnegative) harmonic function u on X such that $v(r) \geqq \int_{B_r(x)} u^p dm$ ([K1]).

Note that assertion a) of Theorem 1 together with the following Lemma imply that every solution $u \in L^p(X, m)$, $p \in]1, \infty[$, of the equation $Lu = 0$ on X is constant (without restriction on the sign of u).

Lemma 2. a) *Let $\eta \in \mathscr{C}^2(\mathbb{R})$ be a convex function with bounded derivatives. Then for every solution u of $Lu = 0$ on X the function $\eta(u)$ is a subsolution.*

b) *For every solution u of $Lu = 0$ on X the function $|u|$ is a subsolution.*

c) *For every $k \in \mathbb{R}$ and every subsolution u of $Lu = 0$ on X the function $u \vee k$ is a subsolution and for every supersolution u of $Lu = 0$ on X the function $u \wedge k$ is a supersolution.*

d) *For all supersolutions u_1, u_2 of $Lu = 0$ on X the function $u = u_1 \wedge u_2$ is a supersolution of $Lu = 0$ on X.*

Proof. Let $v = \eta(u)$ and for every $\psi \in \mathcal{D}_{\text{comp}}(\mathcal{E})$, $\psi \geqq 0$, let $\phi = \eta'(u) \cdot \psi$. Then

$$\begin{aligned} d\Gamma(\psi, v) = \eta'(u)\, d\Gamma(\psi, u) &= d\Gamma(\eta'(u) \cdot \psi, u) - \psi\, d\Gamma(\eta'(u), u) \\ &= d\Gamma(\phi, u) - \eta''(u) \cdot \psi\, d\Gamma(u, u) \leqq d\Gamma(\phi, u). \end{aligned}$$

Now by assumption the integral of the RHS is 0. This proves part a). Part b) follows from a) by approximation. In order to see c) note that the assertion a) of Lemma 2 holds true not only for solutions but also for supersolutions (or subsolutions) of $Lu = 0$ on X provided one assumes in addition that η is decreasing (or increasing, resp.).

The proof of d) is analogous to that of c). Now we have to use the chain rule for functions η of two variables (see [M]). Let $\beta \in \mathscr{C}_b^1(\mathbb{R}^2)$ be a symmetric, convex function with $|\beta'| \leqq 1$ (think of a smooth approximation of $x \to |x|$) and let

$$\eta : (x_1\, x_2) \to \frac{1}{2}(x_1 + x_2) - \beta(x_1 - x_2) \qquad .$$

(think of a smooth approximation of $(x_1, x_2) \to x_1 \wedge x_2$). Moreover, let u_1, u_2 be supersolutions of $Lu = 0$ on X and let $\psi \in \mathcal{D}_{\text{comp}}(\mathcal{E})$, $\psi \geqq 0$, be an arbitrary test function. Then

$$\begin{aligned} 2d\Gamma(\psi, \eta(u_1, u_2)) &= (1 - \beta'(u_1 - u_2))\, d\Gamma(\psi, u_1) + (1 + \beta'(u_1 - u_2))\, d\Gamma(\psi, u_2) \\ &= d\Gamma(\psi \cdot (1 - \beta'(u_1 - u_2)), u_1) + d\Gamma(\psi \cdot (1 + \beta'(u_1 - u_2)), u_2) \\ &\quad - \psi\, d\Gamma(\beta'(u_1 - u_2), u_1 - u_2) \\ &= d\Gamma(\psi \cdot (1 - \beta'(u_1 - u_2)), u_1) + d\Gamma(\psi \cdot (1 + \beta'(u_1 - u_2)), u_2) \\ &\quad - \psi \cdot \beta''\, d\Gamma(u_1 - u_2, u_1 - u_2) \\ &\geqq d\Gamma(\psi \cdot (1 - \beta'(u_1 - u_2)), u_1) + d\Gamma(\psi \cdot (1 + \beta'(u_1 - u_2)), u_2). \end{aligned}$$

Now by the assumptions on u_1 and u_2, each of the integrals on the RHS is nonnegative. (Note that $\psi \cdot (1 \pm \beta'(u_1 - u_2))$ is a nonnegative test function by the assumption on β.) This proves that the function $\eta(u_1, u_2)$ is a supersolution. The claim thus follows by approximation. □

Lemma 3. a) *Every nonnegative supersolution u of the equation $Lu = 0$ on X is excessive for the semigroup $(T_t)_{t>0}$.*

b) *Every function $u \in L^\infty_{\text{loc}}(X, m)$ which is excessive for the semigroup $(T_t)_{t>0}$ is a supersolution of the equation $Lu = 0$ on X.*

Remark. We emphasize that the analogous statement for subsolutions of the equation $Lu = 0$ and for defective functions for the semigroup $(T_t)_{t>0}$ *is wrong*. This can be seen by combining Theorem 2 and the example from [LS] mentioned in the previous Remark b).

Proof. a) Since increasing limits of excessive functions are again excessive we may restrict ourselves to bounded functions u, say $u \leqq 1$. If in addition we would know that $u \in \mathscr{D}(\mathscr{E})$ then the assertion is essentially contained in [F], Theorem 3.2.1. In the general case, we have to make several approximations.

If $u \geqq 0$ is a supersolution of the equation $Lu = 0$ on X then for every relatively compact, open subset $X_0 \subset X$ the function u (or its restriction to X_0) is also a supersolution of the equation $L_0 u = 0$ on X_0. The operator L_0 is obtained from L by putting Dirichlet boundary conditions on ∂X_0. The associated Dirichlet form $\mathscr{E}_0$ is the closure of $\mathscr{E}$ on $\mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X_0)$.

Let $\psi \in \mathscr{D}(\mathscr{E}_0)$ and put $v = u \wedge \psi$. We want to show that $v \in \mathscr{D}(\mathscr{E}_0)$. In order to see this, let $\{\psi_n\}_n$ be a sequence in $\mathscr{D}(\mathscr{E}_0)$ with compact support in X_0 which satisfies $\psi_n \to \psi$ in $\mathscr{E}_1[.]$-norm and $\psi_n \to \psi$ q.e. on X_0. (Here $d\Gamma_1[f] = d\Gamma(f,f) + f^2 dm$ and $\mathscr{E}_1[f] = \int d\Gamma_1[f]$.) We may in addition assume $\psi_n \leqq \psi$ and $\{\psi_n\}$ being increasing. Put $v_n = u \wedge \psi_n$. Then of course $v_n \in \mathscr{D}(\mathscr{E}_0)$ and $v_n \to v$ q.e. on X_0. Moreover,

$$\begin{aligned}\mathscr{E}_1[v - v_n] &\leqq \mathscr{E}_1[\psi - \psi_n] + \int_{\{\psi_n < u < \psi\}} d\Gamma_1[u - \psi_n] \\ &\leqq 2\mathscr{E}_1[\psi - \psi_n] + \int_{\{\psi_n < \psi\}} d\Gamma_1[u - \psi] \to 0\end{aligned}$$

for $n \to \infty$. That is, $v \in \mathscr{D}(\mathscr{E}_0)$.

Now fix $\alpha > 0$ and $x \in X$ and let $X_k = B_k(x)$ for $k > 0$. Let $\psi_k \in \mathscr{D}(\mathscr{E}_k)$ be the α-equilibrium potential of the ball $B_{k/2}(x)$ in the ball $B_k(x)$ and let $u_k = u \wedge \psi_k$. Then $u_k \in \mathscr{D}(\mathscr{E}_k)$. Moreover, u_k is a supersolution of the equation $(L_k - \alpha) u_k = 0$ in X_k. This follows from the fact that u as well as ψ_k are supersolutions of this equation and that the sheaf of nonnegative supersolutions of this equation is inf-stable (which is just a trivial extension of Lemma 2d)). We are now in a position to cite [F] (Theorem 3.2.1) which states that u_k is α-excessive for the heat semigroup $(T_t^k)_{t>0}$ on X_k (with absorption on the complement). In particular, u_k is α-excessive for the heat semigroup $(T_t^{k'})_{t>0}$ on $X_{k'}$ for every $k' < k$. It follows (by increasing limit argument) that $u = \sup_k u_k$ is α-excessive for the heat semigroup $(T_t^k)_{t>0}$ on X_k. Since this holds true for all $k > 0$ and all $\alpha > 0$ we finally obtain that u is excessive for the heat semigroup $(T_t)_{t>0}$ on X.

b) Now let u be a locally bounded excessive function. For a fixed relatively compact, open set $X_0 \subset X$ and $\alpha > 0$ let $\psi \in \mathscr{D}(\mathscr{E})$ be the α-equilibrium potential of X_0 in X. Then u as well as ψ are α-excessive for the semigroup $(T_t)_{t>0}$. By assumption, u is bounded on X_0, say $u \leqq 1$ on X_0.

Let $v = u \wedge \psi$. Then also v is α-excessive and of course $v \leqq \psi$. Since $\psi \in \mathscr{D}(\mathscr{E})$ this implies $v \in \mathscr{D}(\mathscr{E})$ ([F], Lemma 3.3.2). We therefore may apply Theorem 3.2.1 of [F] to conclude that v is a supersolution of the equation $(L - \alpha)v = 0$ on X. But on X_0 the functions u and v coincide. Hence, u is a supersolution of the equation $(L - \alpha)u = 0$ on X_0. Since X_0 and α are arbitrary, this proves the claim. □

For later use, we state the following minimum principle.

Lemma 4. *Let Y be a compact subset of X and let u be a supersolution of the equation $Lu = 0$ on X. If $u \geqq 0$ on $X \setminus Y$ then $u \geqq 0$ on X.*

Proof. Without restriction, we may assume $u \leqq 0$ and $u \equiv 0$ on $X \setminus Y$ (otherwise take $u \wedge 0$). But then obviously $u \in \mathscr{D}(\mathscr{E})$. Therefore, Theorem 3.2.1 of [F] applies which states that u is excessive, in particular, that u is nonnegative. This proves that $u \equiv 0$. □

Proof of Theorem 1. Let $p \in]-\infty, \infty[$, $0 \neq p \neq 1$, and $u \geqq 0$ satisfying (2.2). In the case $p > 1$ we assume that u is a subsolution of $Lu = 0$ on X and in the case $p < 1$ we assume that u is a supersolution. Let $\psi \in \mathscr{D}_{\text{comp}}(\mathscr{E}) \cap L^\infty(X, m)$ be a cut-off function with $d\Gamma(\psi, \psi) \leqq dm$ and for $n \in \mathbb{N}$ let $u_n = \left((u \wedge n) \vee \frac{1}{n}\right)$ and $\phi_n = u_n^{p-1} \cdot \psi^2$. Let Y be any closed set such that ψ is constant on $X \setminus Y$. Then by the chain rule and the Cauchy-Schwarz inequality

$$
\begin{aligned}
(2.3) \qquad & (p-1) \cdot \int d\Gamma(\phi_n, u_n) \\
& = 4(1 - 1/p)^2 \int \psi^2 d\Gamma(u_n^{p/2}, u_n^{p/2}) + 4(1 - 1/p) \int \psi \cdot u_n^{p/2} d\Gamma(\psi, u_n^{p/2}) \\
& \geqq 4(1 - 1/p)^2 \int \psi^2 d\Gamma(u_n^{p/2}, u_n^{p/2}) \\
& \quad - \Big[4 \int_Y u_n^p d\Gamma(\psi, \psi)\Big]^{1/2} \cdot \Big[4(1 - 1/p)^2 \int_Y \psi^2 d\Gamma(u_n^{p/2}, u_n^{p/2})\Big]^{1/2}.
\end{aligned}
$$

Using the assumption $(p-1) \int d\Gamma(\phi_n, u) \leqq 0$, the LHS of (2.3) can be estimated as follows

$$(p-1) \int d\Gamma(\phi_n, u_n) \leqq (p-1) \int d\Gamma(\phi_n, u_n - u).$$

By the truncation property, it suffices to integrate on the RHS over the sets $\{u_n - u < 0\}$ and $\{u_n - u > 0\}$. On each of these sets, the function u_n is constant. Therefore,

$$(p-1) \int d\Gamma(\phi_n, u_n - u) = (p-1) \int_{\{u \neq u_n\}} d\Gamma(\phi_n, -u) = -2(p-1) \int_{\{u \neq u_n\}} \psi \cdot u_n^{p-1} d\Gamma(\psi, u).$$

Splitting up the last term by means of the Cauchy-Schwarz inequality (with arbitrary $\varepsilon > 0$) finally yields

$$(2.4) \quad (p-1) \int d\Gamma(\phi_n, u_n) \leqq \varepsilon (p-1)^2 \int \psi^2 \cdot u_n^{p-2} d\Gamma(u, u) + \frac{1}{\varepsilon} \int_{\{u \neq u_n\}} u_n^p d\Gamma(\psi, \psi)$$

which for $n \to \infty$ tends to $\varepsilon(p-1)^2 \int \psi^2 \cdot u^{p-2} d\Gamma(u, u) = 4\varepsilon(1 - 1/p)^2 \int \psi^2 d\Gamma(u^{p/2}, u^{p/2})$ (for fixed but arbitrary $\varepsilon > 0$). Hence, in the limit $n \to \infty$ (2.3) yields

$$(2.5) \quad \Big[(1 - 1/p)^2 \int \psi^2 d\Gamma(u^{p/2}, u^{p/2})\Big]^2 \leqq \int_Y u^p d\Gamma(\psi, \psi) \cdot (1 - 1/p)^2 \int_Y \psi^2 d\Gamma(u^{p/2}, u^{p/2}).$$

Let us now fix a point $x \in X$ and choose

$$\psi = \varrho_{x,R} \wedge (R - r)$$

and $Y = \bar{B}_R(x) \setminus B_r(x)$ for $0 < r < R$ (cf. (1.5)). Let

$$v(r) = \int_{B_r(x)} u^p dm \quad \text{and} \quad F(r) = (1 - 1/p)^2 \int_{B_r(x)} d\Gamma(u^{p/2}, u^{p/2})$$

and put

$$K = (1 - 1/p)^2 \int_Y \psi^2 d\Gamma(u^{p/2}, u^{p/2}).$$

Then (2.5) implies

$$(2.6)\quad v(R) - v(r) \geqq \int_Y u^p d\Gamma(\psi, \psi) \geqq \frac{[(1-1/p)^2 \int_Y \psi^2 \Gamma(u^{p/2}, u^{p/2})]^2}{(1-1/p)^2 \int_Y \psi^2 \Gamma(u^{p/2}, u^{p/2})}$$

$$= \frac{[F(r) \cdot (R-r)^2 + K]^2}{K} = [F(r) \cdot (R-r)^2] \cdot \left[\frac{F(r) \cdot (R-r)^2}{K} + 1\right]$$

$$\geqq F(r) \cdot (R-r)^2 \cdot \left[\frac{F(r)}{F(R) - F(r)} + 1\right] = \frac{F(r) \cdot F(R)}{F(R) - F(r)} \cdot (R-r)^2 .$$

That is,

$$(2.7)\qquad \frac{1}{F(r)} - \frac{1}{F(R)} \geqq \frac{(R-r)^2}{v(R) - v(r)}$$

for $0 < r < R$. For fixed $r > 0$ take $R_k = 2^k \cdot r$ ($k \in \mathbb{N}$). Then from (2.7) we obtain

$$(2.8)\qquad \frac{1}{F(r)} \geqq \frac{1}{F(R_n)} + \sum_{k=0}^{n-1} \frac{R_k^2}{v(R_{k+1}) - v(R_k)} \geqq \frac{1}{4} \sum_{k=1}^{n} \frac{R_k^2}{v(R_k)} .$$

The assumption (2.2), however, implies $\sum_{k=1}^{\infty} \frac{R_k^2}{v(R_k)} = \infty$. Hence, (2.8) states that $d\Gamma(u^{p/2}, u^{p/2}) \equiv 0$ on $B_r(x)$. Since $r > 0$ was arbitrary this proves that

$$(2.9)\qquad d\Gamma(u^{p/2}, u^{p/2}) \equiv 0 \quad \text{on } X$$

and (since $\mathscr{E}$ was assumed to be irreducible), therefore, u must be constant on X. This proves the claim if $0 \neq p \neq 1$.

The case $p = 0$ can be reduced to the case $p \in]0, 1[$ by the following consideration. Let u be a nonnegative supersolution of the equation $Lu = 0$ on X and let $v(r) = m(B_r(x))$ satisfy (2.2). Then for every $\lambda \in \mathbb{R}$ also the function $u_\lambda = u \wedge \lambda$ is a nonnegative supersolution of the equation $Lu_\lambda = 0$ on X (Lemma 2) and the function $v_\lambda(r) = \int_{B_r(x)} u_\lambda^p dm \leqq \lambda^p \cdot v(r)$ satisfies (2.2). According to our previous results this implies that u_λ is constant (for every $\lambda \in \mathbb{R}$), hence, u is constant. □

Proof of Theorem 2. By Lemma 3 every nonnegative supersolution is excessive. Hence, let $u \in L^1(X, m)$ be either excessive or defective. Since $\mathscr{E}$ is conservative we have $\int_X u\,dm = \int_X T_t u\,dm$ for all $t > 0$. Together with the fact that $T_t u \leqq u$ (resp. "$\geqq$") for all $t > 0$ this implies

$$(2.10)\qquad T_t u \equiv u \quad \text{for all } t > 0,$$

that is, u is invariant for $(T_t)_{t>0}$. From (2.10) it follows that also the function $u_\lambda = u \wedge \lambda$ is excessive (for every $\lambda \in \mathbb{R}$) and by the preceding we see that it is actually invariant. This in turn implies that the function $u^\lambda = u - u_\lambda$ is excessive (and even invariant). The sets $\{u^\lambda > 0\} = \{u > \lambda\}$ are therefore invariant for $\mathscr{E}$ which by assumption is irreducible. Hence, either $u > \lambda$ or $u \leqq \lambda$ and (since λ was arbitrary) finally $u \equiv$ constant. □

3. Recurrence, conservativeness and exponential instability

We investigate three global properties of the Dirichlet form $\mathscr{E}$ which have to do with its "behaviour at ∞". For each of these properties we give a sharp sufficient condition in terms of the volume growth $r \mapsto m(B_r(x))$.

Theorem 3. *Let $v(r) = m(B_r(x))$ be the volume of balls centered at some point $x \in X$ (arbitrary but fixed). If*

$$\int_1^\infty \frac{r}{v(r)}\, dr = \infty \tag{3.1}$$

then $\mathscr{E}$ is recurrent. That is, the following equivalent properties hold true:

a) *Every supersolution $u \geqq 0$ of the equation $Lu = 0$ is constant.*

b) *Every supersolution $u \in L^\infty(X, m)$ of the equation $Lu = 0$ is constant.*

c) *Every subsolution $u \in L^\infty(X, m)$ of the equation $Lu = 0$ is constant.*

d) *Every excessive function for the semigroup $(T_t)_{t>0}$ is constant.*

e) *Every strict potential $Gf = \int_0^\infty T_t f dt$ with $f \in L^1(X, m), f > 0$, is somewhere $+\infty$.*

f) *Every potential $Gf = \int_0^\infty T_t f dt$ with $f \geqq 0, \neq 0$, is identically $+\infty$.*

Proof. The fact that (3.1) implies a) is already contained in part a) of Theorem 1 (case "$p = 0$"). The equivalence of a), b) and c) follows from Lemma 2. For the equivalence of d), e) and f) we refer to [F] and [O1]. Finally, the equivalence of a) and d) follows from Lemma 3. □

Theorem 4. *Let $v(r) = m(B_r(x))$ be the volume of balls centered at some point $x \in X$ (arbitrary but fixed). If*

$$\int_1^\infty \frac{r}{\log v(r)}\, dr = \infty \tag{3.2}$$

then $\mathscr{E}$ is conservative. That is, the following equivalent properties hold true:

a) $T_t 1 = 1$ *for some* $t > 0$.

b) $T_t 1 = 1$ *for all* $t > 0$.

c) *For some* $\alpha > 0$ *every nonnegative solution* $u \in L^\infty(X, m)$ *of the equation* $(L - \alpha) u = 0$ *is identically* 0.

d) *For all* $\alpha > 0$ *every nonnegative subsolution* $u \in L^\infty(X, m)$ *of the equation* $(L - \alpha) u = 0$ *is identically* 0.

Proof. The equivalence a) $\Leftrightarrow$ b) and the implication d) $\Rightarrow$ c) are obvious.

For the proof of the implication c) $\Rightarrow$ b), let $u = 1 - \alpha \cdot G_\alpha 1$ (that is, $u(x) = 1 - \alpha \cdot \int_0^\infty e^{-\alpha t} T_t 1(x) dt$ or, in other words, $u(x) = \mathbb{E}^x[\exp - \alpha \cdot \tau]$ for m-a.e. $x \in X$ with τ being the life time of the process $\mathbb{P}^x$ associated with $\mathscr{E}$). Then $0 \leqq u \leqq 1$ and u is a solution of the equation $(L - \alpha) u = 0$ on X. To see the latter property, note that $(L - \alpha) 1 = -\alpha$ on X and that $G_\alpha 1 = G_\alpha^Y 1 + H_\alpha^Y G_\alpha 1$ for every relatively compact open set $Y \subset X$ (Dynkin's formula) where the first term on the RHS is a solution of the equation $(L - \alpha) v = -1$ in Y (by definition) and the second term is a solution of the equation $(L - \alpha) v = 0$ in Y ([F], Theorem 3.3.4). Hence, u is actually a solution of the equation $(L - \alpha) v = 0$ on X.

Now let us turn to the proof of the implication b) $\Rightarrow$ d). Let $T_t 1 = 1$ for all $t > 0$ and let $0 \leqq u \leqq 1$ be a subsolution of $(L - \alpha) u = 0$ on X for some $\alpha > 0$. Let Y be a compact subset of X and let $v = H_\alpha^Y 1$ be the α-reduced function of the constant 1 on the set $X \setminus Y$. Then $0 \leqq v \leqq 1$ on X and $v = 1$ on $X \setminus Y$. Moreover, v is a supersolution of the equation $(L - \alpha) v = 0$ on X. By the minimum principle (Lemma 4) we therefore obtain $u \leqq v$ on X. On the other hand, the assumption $T_t 1 = 1$ for all $t > 0$ (together with Dynkin's formula) implies

$$v = H_\alpha^Y 1 = \alpha \cdot H_\alpha^Y G_\alpha 1 = \alpha \cdot G_\alpha 1 - \alpha \cdot G_\alpha^Y 1 = 1 - \alpha \cdot G_\alpha^Y 1 .$$

That ist,

$$0 \leqq u \leqq 1 - \alpha \cdot G_\alpha^Y 1$$

for all compact subsets $Y \subset X$ and thus $0 \leqq u \leqq 1 - \alpha \cdot G_\alpha 1 = 0$.

In order to see that (3.2) implies d), we closely follow the argumentation by A. A. Grigor'yan [G1]. Let $u \in L^\infty(X, m)$ be a nonnegative subsolution of the equation $(L - \alpha) u = 0$ on X (for a fixed but arbitrary $\alpha > 0$). That is, $u \in \mathscr{D}_{\text{loc}}(\mathscr{E})$ and

$$\mathscr{E}(u, \phi) + \alpha \int u \cdot \phi \, dm \leqq 0$$

for all nonnegative $\phi \in \mathscr{D}_{\text{comp}}(\mathscr{E})$. Now consider the function $w : \mathbb{R} \to \mathscr{D}_{\text{loc}}(\mathscr{E})$ with $w(t) = u \cdot e^{\alpha \cdot t} - \|u\|_\infty$. It obviously satisfies

$$\text{(3.3)} \qquad \int_{\tau_1}^{\tau_2} \left[\mathscr{E}(w(t), \phi(t)) + \int_X \frac{\partial}{\partial t} w(t) \cdot \phi(t) dm \right] dt \leqq 0$$

for every $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_1 < \tau_2$ and every nonnegative continuous function $\phi : [\tau_1, \tau_2] \to \mathcal{D}_{\text{comp}}(\mathcal{E})$. Moreover, if for m-almost every $x \in X$ the function $t \mapsto \phi(t)(x)$ is piecewise differentiable with derivative $\frac{\partial}{\partial t}\phi(.)(x)$ then (3.3) is equivalent to

$$(3.4)\quad \int_{\tau_1}^{\tau_2}\left[\mathcal{E}(w(t), \phi(t)) - \int_X w(t)\cdot\frac{\partial}{\partial t}\phi(t)\,dm\right]dt + \left[\int_X w(t)\cdot\phi(t)\,dm\right]_{\tau_1}^{\tau_2} \leqq 0.$$

Now let us fix $r > 0$ and for $k \in \mathbb{N}$ let $R_k = 2^k \cdot r$ and $t_k = \frac{9}{64}\sum_{n=1}^{k}\frac{R_n^2}{\log v(R_n)}$. Assumption (3.2) implies that $\lim_{k\to\infty} t_k = \infty$, that is, for every $T > 0$ there exists $N \in \mathbb{N}$ with $t_N \geqq T$. Furthermore, fix $x \in X$ and let $\phi_k = w_+ \cdot \psi_k^2 \cdot g_k$ where $\psi_k(t, y) = \left(\frac{2}{R_k}\cdot\varrho_{x,2R_k}(y)\right) \wedge 1$ and $g_k(t, y) = \exp\left(-\frac{(\varrho_x - R_k)_+^2}{4(T - t_k - t)}\right)$.

Adding up (3.3) and (3.4) we get with the above function $\phi = \phi_k$

$$\begin{aligned}
&\int_X w(\tau_2)\cdot\phi_k(\tau_2)\,dm - \int_X w(\tau_1)\cdot\phi_k(\tau_1)\,dm \\
&\leqq -2\int_{\tau_1}^{\tau_2}\mathcal{E}(w, \phi)\,dt + \int_{\tau_1}^{\tau_2}\int_X w\cdot\frac{\partial}{\partial_t}\phi\,dm\,dt - \int_{\tau_1}^{\tau_2}\int_X \frac{\partial}{\partial_t} w\cdot\phi\,dm\,dt \\
&= -2\int_{\tau_1}^{\tau_2}\int \psi_k^2\cdot g_k\,d\Gamma(w_+, w_+)\,dt - 4\int_{\tau_1}^{\tau_2}\int \psi_k\cdot w_+\cdot g_k\,d\Gamma(w_+, \psi_k)\,dt \\
&\quad -2\int_{\tau_1}^{\tau_2}\int w_+\cdot\psi_k^2\cdot g_k\cdot\frac{-2(\varrho_x - R_k)_+}{4(T - t_k - t)}\,d\Gamma(w_+, \varrho_x)\,dt \\
&\quad +\int_{\tau_1}^{\tau_2}\int \psi_k^2\cdot w_+^2\cdot g_k\cdot\frac{(\varrho_x - R_k)_+^2}{4(T - t_k - t)^2}\,dm\,dt
\end{aligned}$$

where we have used Leibniz and chain rules and the fact that ψ does not depend on t. Each of the integrals on the RHS containing $d\Gamma(w_+, \psi_k)$ or $d\Gamma(w_+, \varrho_x)$ can be splitted up by means of Cauchy-Schwarz inequality into an integral $+\int\int \psi_k^2\cdot g_k\,d\Gamma(w_+, w_+)\,dt$ and an integral containing $d\Gamma(\psi_k, \psi_k)$ or $d\Gamma(\varrho_x, \varrho_x)$, respectively. This yields

$$\begin{aligned}
\text{RHS} &\leqq 4\int_{\tau_1}^{\tau_2}\int w_+^2\cdot g_k\,d\Gamma(\psi_k, \psi_k)\,dt \\
&\quad - \int_{\tau_1}^{\tau_2}\int w_+^2\cdot\psi_k^2\cdot g_k\,\frac{(\varrho_x - R_k)_+^2}{4(T - t_k - t)^2}\,d(\Gamma(\varrho_x, \varrho_x) - m)\,dt \\
&\leqq \frac{16}{R_k^2}\int_{\tau_1}^{\tau_2}\int_{B_{2R_k}\setminus B_{3/2R_k}} w_+^2\cdot g_k\,dm\,dt
\end{aligned}$$

where in the last inequality we have used that $d\Gamma(\varrho_x, \varrho_x) \leqq dm$ on X and $d\Gamma(\psi_k, \psi_k) \leqq 4/R_k^2$ on $B_{2R_k}\setminus B_{3/2R_k}$ and $= 0$ elsewhere. Choosing now $\tau_2 = T - t_k$ and $\tau_1 = T - t_{k+1}$ we finally obtain

$$\begin{aligned}
&\int_{B_{R_k}} w_+^2(T-t_k)\,dm - \int_{B_{R_{k+1}}} w_+^2(T-t_{k+1})\,dm \\
&\leqq \int_X w_+^2(T-t_k)\cdot\psi_k^2\cdot g_k(T-t_k)\,dm - \int_X w_+^2(T-t_{k+1})\cdot\psi_k^2\cdot g_k(T-t_{k+1})\,dm \\
&\leqq \frac{16}{R_k^2}\int_{T-t_{k+1}}^{T-t_k}\int_{B_{2R_k}\setminus B_{3/2R_{k+1}}} w_+^2(t)\cdot g_k(t)\,dm\,dt \\
&\leqq \frac{16}{R_k^2}\cdot v(2R_k)\cdot\|u\|_\infty^2\cdot\exp(\alpha T)\cdot\exp\left(-\frac{(3/2R_k)^2}{4(t_{k+1}-t_k)}\right)\leqq\frac{C}{R_k^2}
\end{aligned}$$

(with $C = 8\|u\|_\infty^2$) by the choice of t_k. That is,

$$\int_{B_{R_k}} w_+^2(T-t_k)\,dm \leqq \int_{B_{R_{k+1}}} w_+^2(T-t_{k+1})\,dm + \frac{C}{R_k^2}$$

and thus

$$\int_{B_r} w_+^2(T)\,dm \leqq \int_{B_{R_N}} w_+^2(T-t_N)\,dm + \sum_{k=0}^{N-1}\frac{C}{R_k^2} \leqq \frac{4}{3}\cdot\frac{C}{r^2} \tag{3.5}$$

since $T-t_N \leqq 0$ and $w(t)\leqq 0$ for $t \leqq 0$ by definition. Letting r tend to ∞, (3.5) implies $w(T)\leqq 0$. Since $T>0$ is arbitrary, we obtain that u must vanish. □

Remarks. a) Note that the recurrence condition (3.1) is of course satisfied if

$$m(B_{r_n}(x)) \leqq C\cdot r_n^2 \tag{3.6}$$

(but also if $m(B_{r_n}(x)) \leqq C\cdot r_n^2\cdot\log r_n$) for a point $x\in X$, sequence $r_n\to\infty$ and a constant C.

Similarly, the conservativeness condition (3.2) is satisfied whenever

$$m(B_{r_n}(x)) \leqq e^{C\cdot r_n^2} \tag{3.7}$$

(or $m(B_{r_n}(x)) \leqq r_n^{C\cdot r_n^2}$) for a point $x\in X$, a sequence $r_n\to\infty$ and a constant C.

b) For general Dirichlet forms, necessary and sufficient conditions for recurrence respectively conservativeness have been given by Y. Oshima [O2]. In the diffusion case, a sufficient condition for conservativeness (3.7) has been derived by M. Takeda [T]. Using our Lemma 1 one can show that Takeda's condition is essentially equivalent to (3.7). The slightly weaker condition (3.2) has the advantage of being sharp (cf. Remark a) after Cor. 2).

For the canonical Dirichlet form on a complete Riemannian manifold, the statements of Theorems 3 and 4 are well-known. The recurrence criteria (3.6) and (3.1) are due to S.Y. Cheng and S.T. Yau [CY] and to L. Karp [K1]. The conservativeness criteria (3.7) and (3.2) are due to L. Karp and P. Li [KL] and to A.A. Grigor'yan [G1]. They improve previous conditions in terms of the (Ricci) curvature.

For recurrence or conservativeness conditions in the case of elliptic differential operators on $\mathbb{R}^N$, we refer to E.B. Davies [D2], K. Ichihara [I]. Y. Oshima [O2] and M. Takeda [T].

c) In [S2], we prove that condition (3.1) is *sufficient and necessary* for recurrence provided we require in addition that a uniform parabolic Harnack inequality holds true for $\mathscr{E}$. The latter property turns out to be equivalent to the validity of a uniform Poincaré inequality and a doubling property (cf. [S2]). For the canonical Dirichlet form on a complete Riemannian manifold these properties are satisfied provided the Ricci curvature is nonnegative. In this situation, the equivalence of recurrence and property (3.1) was proved by N. Varopoulos [V].

If $\mathscr{E}$ is conservative then obviously $\|T_t\|_{1,1} = \|T_t\|_{\infty,\infty} = 1$ for all $t > 0$. According to Theorem 4 this is for instance the case if

$$\liminf_{r\to\infty} \frac{1}{r^2} \log v(r) < \infty .$$

Our next result concerns the long time behaviour of $\|T_t\|_{p,p}$. The semigroup $(T_t)_{t>0}$ is called *exponentially stable* on $L^p(X, m)$ if $\|T_t\|_{p,p} \to 0$ for $t \to \infty$ or, equivalenty, if $\|T_t\|_{p,p} < 1$ for some $t > 0$. We shall see that the condition

$$\liminf_{r\to\infty} \frac{1}{r} \log v(r) = 0$$

implies $\|T_t\|_{2,2} = 1$ for some $t > 0$ and hence $\|T_t\|_{p,p} = 1$, for all $p \in [1, \infty]$ and $t > 0$.

Let λ be the infimum of the spectrum of the positive semidefinite, selfadjoint operator $-L$ on $L^2(X, m)$. By spectral theory, $\|T_t\|_{2,2} = \exp(-\lambda \cdot t)$ for all $t > 0$. The L^2-spectral bound λ can be calculated as Rayleigh-Ritz quotient by the formula

$$\lambda = \inf \left\{ \frac{\mathscr{E}(u, u)}{\int u^2\, dm} : u \in \mathscr{D}(\mathscr{E}),\ \int u^2\, dm \neq 0 \right\}.$$

Theorem 5. *Let $v_*(r) = \inf_{x\in X} m(B_r(x))/m(B_1(x))$ be the lower bound for the volume growth of balls centered at variable points. If*

$$\liminf_{r\to\infty} \frac{1}{r} \log v_*(r) \leqq k \tag{3.8}$$

then the L^2-spectral bound λ satisfies $0 \leqq \lambda \leqq \frac{k^2}{4}$.

Condition (3.8) is of course satisfied if $\liminf_{r\to\infty} \frac{1}{r} \log v(r) \leqq k$ with $v(r) = m(B_r(x))$ for some fixed $x \in X$.

Proof. Let $v(R, x) = m(B_R(x))$ be the volume of the ball $B_R(x)$. For $\alpha > k/2$ and $\varepsilon > 0$ there exists a number $R > 1$ and a point $x \in X$ with $\frac{v(R,x)\cdot \exp(-2\alpha\cdot R)}{v(1,x)\cdot\exp(-2\alpha)} \leqq \frac{\varepsilon^2}{1-\varepsilon}$. Let us fix x and R and consider the function $u : y \mapsto (e^{-\alpha\cdot\varrho_x(y)} - e^{-\alpha\cdot R})_+$ which (according to

Lemma 1) satisfies $d\Gamma(u,u) \leqq \alpha^2 \cdot e^{-2\alpha \cdot \varrho_x} dm$ (and, obviously, $\equiv 0$ in $X \setminus B_R(x)$) and which therefore lies in $\mathscr{D}(\mathscr{E})$. Then

$$\frac{1}{\lambda} \geqq \frac{\int u^2 dm}{\mathscr{E}(u,u)} \geqq \frac{\int_{B_R} (\exp(-\alpha \cdot \varrho_x) - \exp(-\alpha \cdot R))^2 dm}{\alpha^2 \int_{B_R} \exp(-2\alpha \cdot \varrho_x) dm}$$

$$\geqq (1-\varepsilon) \frac{\int_{B_R} \exp(-2\alpha \cdot \varrho_x) dm}{\alpha^2 \int_{B_R} \exp(-2\alpha \cdot \varrho_x) dm} - (1/\varepsilon - 1) \frac{\int_{B_R} \exp(-2\alpha \cdot R) dm}{\alpha^2 \int_{B_R} \exp(-2\alpha \cdot \varrho_x) dm}$$

$$\geqq \frac{1-\varepsilon}{\alpha^2} - \frac{1-\varepsilon}{\varepsilon \cdot \alpha^2} \cdot \frac{v(R,x) \cdot \exp(-2\alpha \cdot R)}{v(1,x) \cdot \exp(-2\alpha)} \geqq \frac{1-2\varepsilon}{\alpha^2}.$$

This proves the claim. □

Corollary 2. *If the volume $v(r) = m(B_r(x))$ of balls centered at some point $x \in X$ (arbitrary but fixed) grows subexponentially, i.e. if*

(3.9) $$\liminf_{r \to \infty} \frac{1}{r} \log v(r) = 0,$$

(or, more generally, if $\liminf_{r\to\infty} v_*(r) = 0$ with $v_*(r) = \inf_{x \in X} m(B_r(x))/m(B_1(x))$) then the semigroup $(T_t)_{t>0}$ is exponentially instable on each of the spaces $L^p(X,m)$, $p \in [1,\infty]$. That is, the following equivalent properties hold true:

a) $\|T_t\|_{2,2} = 1$ *for some* $t > 0$.

b) $\|T_t\|_{p,p} = 1$ *for all* $t > 0$ *and all* $p \in [1,\infty]$.

c) *The bottom of the spectrum of* $-L$ *is* 0.

d) $\inf\{\mathscr{E}(u,u) : u \in \mathscr{D}(\mathscr{E}), \int u^2 dm = 1\} = 0$.

Remarks. a) The assumptions in Theorems 3, 4 and 5 are sharp in the following sense: given a smooth function v (with $v, v' \geqq 0$) which does not satisfy (3.1) (or (3.2) or (3.8)) then there exists a complete Riemannian manifold X with $v(r) \geqq m(B_r(x))$ for some $x \in X$ such that the canonical heat semigroup on X satisfies $\int_0^\infty T_t dt \not\equiv \infty$ (or $T_t 1 \not\equiv 1$ or $\|T_t\|_{2,2} < \exp\left(-\frac{k^2}{4} t\right)$, resp.). See [K1] for (3.1), [G2] for (3.2) and [B] for (3.8).

b) On Riemannian manifolds, the volume $v(r)$ of the ball $B_r(x)$ (for a fixed point $x \in X$) can be estimated in terms of a lower bounded $K(r)$ for the Ricci curvature on $B_r(x)$ according to:

(3.10) $$v(r) \leqq C \cdot r^N \cdot \exp\left(\sqrt{(N-1) \cdot K(r)} \cdot r\right)$$

where N denotes the dimension of X and C is a constant depending on x (Bishop's comparison theorem, cf. [Sa]). Conditions in term of the Ricci curvature, however, are not stable under quasi-isometric changes of the Riemannian metric whereas conditions on the volume growth are stable (cf. following remark).

c) Each of the conditions (3.1), (3.2) and (3.8) on the volume growth $r \mapsto m(B_r(x))$ of balls derived from a Dirichlet form $\mathscr{E}$ on $L^2(X, m)$ is satisfied if and only if the same condition is satisfied for the volume growth $\tilde{m}(\tilde{B}_r(x))$ of balls derived from another Dirichlet form $\tilde{\mathscr{E}}$ on $L^2(X, \tilde{m})$ with the property that for some $k \in [1, \infty[$

$$\frac{1}{k} \cdot \mathscr{E}(u, u) \leqq \tilde{\mathscr{E}}(u, u) \leqq k \cdot \mathscr{E}(u, u) \tag{3.11}$$

(for all $u \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\tilde{\mathscr{E}})$) and

$$\frac{1}{k} \cdot m \leqq \tilde{m} \leqq k \cdot m. \tag{3.12}$$

In order to see this, let us mention that (3.11) implies $\frac{1}{k} \cdot \Gamma(u, u) \leqq \tilde{\Gamma}(u, u) \leqq k \cdot \Gamma(u, u)$ which together with (3.12) implies $B_{r/k}(x) \subset \tilde{B}_r(x) \subset B_{kr}(x)$ and hence

$$\frac{1}{k} \cdot m(B_{r/k}(x)) \leqq \tilde{m}(\tilde{B}_r(x)) \leqq k \cdot m(B_{kr}(x)).$$

For the "if"-part in the above assertion it suffices that (3.12) and the upper inequality in (3.11) are satisfied.

d) It is worthwhile to mention that recurrence as well as exponential instability is preserved under quasi-isometric changes (i.e. changes of $\mathscr{E}$ and m satisfying (3.11) and (3.12)). In the case of exponential instability, this is immediate from the definition of λ as a Rayleigh-Ritz quotient. In the case of recurrence, this is an obvious corollary of Y. Oshima's recurrence criterion [O2].

In contrast to that, conservativeness is *not* preserved under quasi-isometric changes, see T. Lyons [L2].

4. Appendix

4.1. The energy measure. In this section we will give a brief survey on properties of the energy measure Γ. Almost all of these properties have already been stated by M. Fukushima [F], Y. Lejan [Le] and U. Mosco [M]. Here we only give proofs of these properties which are not stated previously.

We recall that for $u \in \mathscr{D}(\mathscr{E}) \cap L^\infty(X, m)$ the Radon measure $\Gamma(u, u)$ is defined by the formula

$$\int_X \phi \, d\Gamma(u, u) = \mathscr{E}(u, \phi u) - \frac{1}{2} \mathscr{E}(u^2, \phi)$$

for every $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, m)$ and every $\phi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(X)$. Alternatively, it can be defined by the formula

$$\int_X \phi \, d\Gamma(u, u) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X \phi(x) \cdot [u(x) - u(y)]^2 \, T_t(x, dy) \, m(dx)$$

for every $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, m)$ and every $\phi \in \mathcal{C}_0(X)$. For $u \in \mathcal{D}(\mathcal{E})$ the measure $\Gamma(u, u)$ is obtained as the increasing limit of the measures $\Gamma(u_n, u_n)$ with $u_n = (u \wedge n) \vee (-n)$. These are (nonnegative) finite Radon measures on X which do not charge exceptional sets (i.e. sets of capacity zero). From the quadratic form $u \mapsto \mathcal{E}(u, u)$ on $\mathcal{D}(\mathcal{E})$ one obtains by polarization the symmetric bilinear form $(u, v) \mapsto \Gamma(u, v)$ on $\mathcal{D}(\mathcal{E})$ with values in the signed Radon measures (of finite total variation). These energy measures have the following basic properties.

i) Locality. From the strong locality of $\mathcal{E}$ one obtains the *strong locality* of Γ:

$$1_G \, d\Gamma(u, w) = 0$$

for all functions $u, w \in \mathcal{D}(\mathcal{E})$ and all open sets $G \subset X$ on which u is constant. This immediately implies the following *locality* of Γ:

$$1_G \, d\Gamma(u, w) = 1_G \, d\Gamma(v, w)$$

for all open sets $G \subset X$ and all functions $u, v, w \in \mathcal{D}(\mathcal{E})$ with $u = v$ on G. Both properties extend to arbitrary measurable sets $G \subset X$. In the general case, however, one has to emphasize that the equality $u = v$ must hold (for the quasi-continuous versions) q.e. on G and not only m-a.e. (In order to prove this extension, note that the truncation property stated below implies $1_G \, d\Gamma(u, w) - 1_G \, d\Gamma(v, w) = 1_G \, d\Gamma(u - v, w) = 1_G \cdot 1_{\{u \neq v\}} \, d\Gamma(u - v, w)$ and that the RHS vanishes if $\{u \neq v\} \cap G$ is exceptional.)

The locality of Γ allows to extend its definition to the set $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ being the set of all m-measurable functions u on X for which for every compact set $Y \subset X$ there exists a function $u' \in \mathcal{D}(\mathcal{E})$ with $u = u'$ m-a.e. on Y. For $u, v \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ the signed Radon measure $\Gamma(u, v)$ will be defined via its restriction to relatively compact open sets $G \subset X$ by $1_G \, d\Gamma(u, v) = 1_G \, d\Gamma(u', v')$ where u' and v' are suitably chosen functions in $\mathcal{D}(\mathcal{E})$ which coincide on G with u and v, respectively.

ii) Cauchy-Schwarz inequality. For $u, v \in \mathcal{D}(\mathcal{E})$ and $f, g \in L^\infty(X, m)$ we have the estimates

$$\begin{aligned} \int_X fg \, d\Gamma(u, v) &\leqq \Big(\int_X f^2 d\Gamma(u, u)\Big)^{1/2} \cdot \Big(\int_X g^2 d\Gamma(v, v)\Big)^{1/2} \\ &\leqq \frac{1}{2} \int_X f^2 d\Gamma(u, u) + \frac{1}{2} \int_X g^2 d\Gamma(v, v). \end{aligned}$$

iii) Continuity. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\mathcal{E})$ with $u_n \to u \in \mathcal{D}(\mathcal{E})$ in the pseudo norm $\sqrt{\mathcal{E}}$. Then

$$\int_X f \, d\Gamma(u - u_n, v) \to 0$$

for all $v \in \mathscr{D}(\mathscr{E})$ and all $f \in L^\infty(X, m)$ (since the modulus of the LHS can be estimated from above by $\|f\|_\infty \cdot \mathscr{E}(u - u_n, u - u_n)^{1/2} \cdot \mathscr{E}(v, v)^{1/2}$). This in turn implies that (under the same assumptions)

$$\int_X f d\Gamma(u_n, u_n) \to \int_X f d\Gamma(u, u).$$

iv) Truncation property. A straightforward consequence of the locality is the *truncation property*:

$$d\Gamma(u \wedge v, w) = 1_{\{u < v\}} d\Gamma(u, w) + 1_{\{u \geqq v\}} d\Gamma(v, w)$$

for all $u, v, w \in \mathscr{D}_{\text{loc}}(\mathscr{E})$. In particular,

$$d\Gamma(u_+, w) = 1_{\{u > 0\}} d\Gamma(u, w) \quad \text{and} \quad d\Gamma(u \wedge v, u \wedge v) = 1_{\{u < v\}} d\Gamma(u, u) + 1_{\{u \geqq v\}} d\Gamma(v, v).$$

(The formula for $d\Gamma(u_+, w)$ was proved in [M]. From this one, the other two formulae can be easily deduced in the following way:

$$\begin{aligned} d\Gamma(u \wedge v, w) &= d\Gamma\big(v - (v - u)_+, w\big) = d\Gamma(v, w) - 1_{\{v > u\}} d\Gamma(v - u, w) \\ &= 1_{\{u < v\}} d\Gamma(u, w) + 1_{\{u \geqq v\}} d\Gamma(v, w) \end{aligned}$$

and

$$d\Gamma(u \wedge v, u \wedge v) = 1_{\{u < v\}} d\Gamma(u, u \wedge v) + 1_{\{u \geqq v\}} d\Gamma(v, u \wedge v) = 1_{\{u < v\}} d\Gamma(u, u) + 1_{\{u \geqq v\}} d\Gamma(v, v).$$

Note, however, that in our first formula the term $1_{\{u = v\}} d\Gamma(u, w)$ occurs which in a similar formula in [BM2] was forgotten.) Obviously, the truncation property also implies the *contraction property*

$$d\Gamma(u^\natural, u^\natural) \leqq d\Gamma(u, u)$$

where $u^\natural = (u \wedge 1) \vee 0$.

v) Leibniz rule. The strong locality of $\mathscr{E}$ (or Γ) can be proven to be equivalent to the validity of the *Leibniz rule*:

$$d\Gamma(u \cdot v, w) = u\, d\Gamma(v, w) + v\, d\Gamma(u, w)$$

for all $u, v \in \mathscr{D}_{\text{loc}}(\mathscr{E}) \cap L^\infty_{\text{loc}}(X, m)$ and $w \in \mathscr{D}_{\text{loc}}(\mathscr{E})$.

vi) Chain rule. Let $\eta \in \mathscr{C}_b^1(\mathbb{R})$ with bounded derivative η'. Then the *chain rule* states that $u \in \mathscr{D}_{\text{loc}}(\mathscr{E})$ implies $\eta(u) \in \mathscr{D}_{\text{loc}}(\mathscr{E})$ and

$$d\Gamma\big(\eta(u), \phi\big) = \eta'(u)\, d\Gamma(u, \phi)$$

for all $\phi \in \mathscr{D}_{\text{loc}}(\mathscr{E}) \cap L^\infty_{\text{loc}}(X, m)$. Obviously, the above formula also holds true for the function $\eta : t \mapsto |t|^p$ provided we restrict ourselves to functions u with ess sup $u < \infty$ in the case $p > 1$ or with ess inf $u > 0$ in the case $p < 1$. We only mention that also a chain rule for functions $\eta \in \mathscr{C}_b^1(\mathbb{R}^k)$ of several variables holds true.

4.2. Assumption (A) and the distance function. This section is devoted to the investigation of the distance ϱ. Also our basic Assumption (A) will be discussed. In this section we do *not assume* a priori that (A) is satisfied. Besides the distance ϱ (defined by (1.3)) we consider the smaller distance

$$(4.1) \quad \varrho^0(x, y) = \sup\{u(x) - u(y) \colon u \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X),\ d\Gamma(u, u) \leqq dm \text{ on } X\}$$

which of course also defines a pseudo metric on X. Instead of requiring (A) we make the weaker

Assumption (A'). *The topology induced by ϱ is equivalent to the original topology on X.*

Under (A'), one always has $0 < \varrho(x, y) < \infty$ whenever $x \neq y$. In order to see the upper inequality, note that (under (A')) for every $x \in X$ the set $X(x) = \bigcup_{r>0} B_r(x)$ is open and closed. Since $\mathscr{E}$ is irreducible and local, X is connected and thus $X(x) = X$.

Asssumption (A') also implies that for every fixed $x \in X$ and sufficiently small $r > 0$ the ball $B_r(x)$ is relatively compact. However, it does not necessarily imply that *all* the balls $B_r(x)$ are relatively compact. In [S3] we prove that under (A') the fact that all balls are relatively compact is equivalent to the completeness of the metric space (X, ϱ). For the canonical Dirichlet form on a Riemannian manifold X, the completeness of (X, ϱ) means that X has no boundary. In [S3] it is also shown that under (A') the boundary of the open ball $B_r(x) = \{y : \varrho(x, y) < r\}$ is given by the sphere $S_r(x) = \{y : \varrho(x, y) = r\}$ (which in general metric spaces might be a much bigger set than the boundary of $B_r(x)$).

All the preceding remarks also apply to ϱ^0 if we assume that (A') is satisfied with ϱ^0 in the place of ϱ.

Lemma 1'. a) *Let ϱ satisfy* (A'). *Then for every $x \in X$ the distance function $\varrho_x : y \mapsto \varrho(x, y)$ on X satisfies $\varrho_x \in \mathscr{D}_{\text{loc}}(\mathscr{E}) \cap \mathscr{C}(X)$ and*

$$(4.2) \qquad d\Gamma(\varrho_x, \varrho_x) \leqq dm \quad \text{on } X.$$

Moreover, for every $r > 0$ the cut-off function $\varrho_{x,r} : y \mapsto (r - \varrho(x, y))_+$ on X also satisfies (4.2) *and $\varrho_{x,r} \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$ if (and only if) $B_r(x)$ is relatively compact.*

b) *The same assertions as in* a) *hold true with ϱ^0 instead of ϱ.*

Proof. a) For every $n \in \mathbb{N}$ there exists a countable number of points $y_i = y_i^{(n)} \in X$, $i \in \mathbb{N}$, such that $\{B_{1/n}(y_i) : i \in \mathbb{N}\}$ is a covering of the space X. For every $i \in \mathbb{N}$ there exists a function $\tilde{\phi}_i = \tilde{\phi}_i^{(n)} \in \mathscr{D}_{\text{loc}}(\mathscr{E}) \cap \mathscr{C}(X)$ with $d\Gamma(\tilde{\phi}_i, \tilde{\phi}_i) \leqq dm$ on X and $\tilde{\phi}_i(x) - \tilde{\phi}_i(y_i) \geqq \varrho(x, y_i) - \frac{1}{n}$. Since $\tilde{\phi}_i$ is one of the admissible functions in the definition of the distance ϱ it also satisfies $\tilde{\phi}_i(y) \geqq \tilde{\phi}_i(x) - \varrho(x, y)$ for all $y \in X$ as well as $\tilde{\phi}_i(y) \leqq \tilde{\phi}_i(y_i) + \frac{1}{n}$ for all $y \in B_{1/n}(y_i)$. Together with the triangle inequality

$\varrho(x, y) \geqq \varrho(x, y_i) - \frac{1}{n}$ for all $y \in B_{1/n}(y_i)$ the latter yields $\tilde{\phi}_i(y) \leqq \tilde{\phi}_i(x) - \varrho(x, y) + \frac{3}{n}$ for all $y \in B_{1/n}(y_i)$.

Let $\phi_i : y \mapsto (\tilde{\phi}_i(x) - \tilde{\phi}_i(y))_+$. Then $\phi_i = \phi_i^{(n)}$ satisfies

(4.3) $$\phi_i \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E}) \cap \mathscr{C}(X) \quad \text{with} \quad d\Gamma(\phi_i, \phi_i) \leqq dm \quad \text{on } X,$$

(4.4) $$0 \leqq \phi_i(y) \leqq \varrho(x, y) \quad \text{for all } y \in X,$$

(4.5) $$\phi_i(y) \geqq \varrho(x, y) - \frac{3}{n} \quad \text{for all } y \in B_{1/n}(y_i).$$

Properties (4.3)–(4.5) remain valid if we replace ϕ_i by $\sup_{1 \leqq j \leqq i} \phi_j$. Hence, we may and will assume from now on that the sequence ϕ_i is also increasing in i.

Now let $X_0 \subset X$ be a relatively compact open set. Since the distance function is by assumption (A′) a continuous function on X, it is bounded on X_0, say $\varrho(x, y) \leqq k$ for $y \in X_0$. Moreover, there exists a function $\psi \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$ with compact support $Y \subset X$ satisfying $0 \leqq \psi \leqq 1$ on X and $\psi \equiv 1$ on X_0 ([F], Lemma 1.4.2). Let $\Phi_i = \Phi_i^{(n)} = \phi_i^{(n)} \wedge (k \cdot \psi)$. Then $\Phi_i \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$ and $\Phi_i = \phi_i$ on X_0.

Consider the family $\tilde{\mathscr{D}} = \{\Phi_i^{(n)} : i \in \mathbb{N}, n \in \mathbb{N}\} \subset \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$. The set $\tilde{\mathscr{D}}$ is uniformly bounded in the norm $\sqrt{\mathscr{E}_1}[.] : \phi \mapsto \sqrt{\mathscr{E}(\phi, \phi) + \int \phi^2 dm}$ according to

$$\begin{aligned} \mathscr{E}(\Phi_i, \Phi_i) + \int \Phi_i^2 dm &\leqq \int_Y d\Gamma(\phi_i, \phi_i) + k^2 \cdot \mathscr{E}(\psi, \psi) + k^2 \int \psi^2 dm \\ &\leqq m(Y) + k^2 \cdot \mathscr{E}(\psi, \psi) + k^2 \int \psi^2 dm < \infty . \end{aligned}$$

Hence, for every sequence in $\tilde{\mathscr{D}}$ there exists a cluster value $\Phi \in \mathscr{D}(\mathscr{E}) \cap L^\infty(X, m)$. Let $\Phi^{(n)} = \lim_{i \to \infty} \Phi_i^{(n)}$. Properties (4.4) and (4.5) imply that $\varrho_x - \frac{3}{n} \leqq \Phi^{(n)} \leqq \varrho_x$ on X_0 and (4.5) implies $d\Gamma(\Phi^{(n)}, \Phi^{(n)}) \leqq dm$ on X_0. Both properties remain valid if we replace $\Phi^{(n)}$ by $\sup_{1 \leqq l \leqq n} \Phi^{(l)}$. Hence, we may and will assume from now on that the sequence $\Phi^{(n)}$ is increasing in n.

The sequence $\{\Phi^{(n)}\}_n$ is again uniformly $\mathscr{E}_1$-bounded. Hence, $\Phi = \lim_{n \to \infty} \Phi^{(n)}$ exists in $\Phi \in \mathscr{D}(\mathscr{E}) \cap L^\infty(X, m)$. From the respective properties of $\Phi^{(n)}$ it follows that $\Phi = \varrho_x$ on X_0 and $d\Gamma(\Phi, \Phi) \leqq dm$ on X_0. That is, $d\Gamma(\varrho_x, \varrho_x) \leqq dm$ on X_0 and (since X_0 was arbitrary) actually

(4.6) $$d\Gamma(\varrho_x, \varrho_x) \leqq dm \quad \text{on } X.$$

The rest is obvious.

b) The proof runs in the same way. Now the functions $\tilde{\phi}_i^{(n)}$ are in $\mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$. The functions $\phi_i^{(n)}$ are again in $\mathscr{D}_{\mathrm{loc}}(\mathscr{E}) \cap \mathscr{C}(X)$. □

The proof of Lemma 1 is inspired by the construction of cut-off functions in [BM2]. An independent proof of Lemma 1 was found by M. Biroli (private communication).

Proposition 1. a) *The distance ϱ satisfies* (A′) *if and only if ϱ^0 satisfies* (A′).

b) *Under* (A′) *for every $x \in X$ and $r > 0$ the ball $B_r(x)$ is relatively compact if and only if the ball $B_r^0(x)$ is relatively compact and in this case $B_r(x) = B_r^0(x)$.*

c) *The distance ϱ satisfies* (A) *if and only if ϱ^0 satisfies* (A) *and in this case $\varrho \equiv \varrho^0$.*

Proof. a) Let ϱ satisfy (A′). Then for fixed $x \in X$ and $\varepsilon > 0$ sufficiently small the ball $B_\varepsilon(x)$ is relatively compact. Lemma 1′ implies that $\varrho_{x,\varepsilon} \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$. Hence, $\varrho_{x,\varepsilon}$ is an admissible function in the definition of the distance $\varrho^0(x, .)$ which proves $\varrho_x^0 \geqq \varrho_x$ in $B_\varepsilon(x)$. Since the converse inequality holds true anyway, we have $B_\varepsilon(x) = B_\varepsilon^0(x)$.

Now assume conversely that ϱ^0 satisfies (A′). Then again for fixed $x \in X$ and $\varepsilon > 0$ sufficiently small the ball $B_\varepsilon^0(x)$ is relatively compact. Let $\psi = \varrho_{x,\varepsilon}^0$. Then for every admissible function u in the definition of ϱ (i.e. $u \in \mathscr{D}_{\text{loc}}(\mathscr{E}) \cap \mathscr{C}(X)$ with $d\Gamma(u, u) \leqq dm$) the function $v = u \wedge \psi$ is an admissible function in the definition of ϱ^0 (i.e. $v \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X)$ with $d\Gamma(v, v) \leqq dm$) because $d\Gamma(v, v) = 1_{\{u \leqq \psi\}} d\Gamma(u, u) + 1_{\{u > \psi\}} d\Gamma(\psi, \psi) \leqq dm$. Hence, $\varrho_x^0 \geqq \varrho_x$ in $B_{\varepsilon/2}^0(x)$. Since always $\varrho^0 \leqq \varrho$ on X this proves that also ϱ satisfies (A′).

b) If $B_r^0(x)$ is relatively compact then also $B_r(x)$ is relatively compact since always $B_r(x) \subset B_r^0(x)$. If $B_r(x)$ is relatively compact then as in part a) it follows that $B_r(x) = B_r^0(x)$.

c) is obvious from a) and b). □

4.3. Irreducibility. Let us close with some comments on the assumption that $\mathscr{E}$ is irreducible. Here we neither require (A) nor (A′). We recall that an m-measurable set $Y \subset X$ is called *invariant for $\mathscr{E}$* if $u \in \mathscr{D}(\mathscr{E})$ implies $1_Y \cdot u \in \mathscr{D}(\mathscr{E})$ and

$$\mathscr{E}(u, u) = \mathscr{E}(1_Y u, 1_Y u) + \mathscr{E}(1_{X \setminus Y} u, 1_{X \setminus Y} u).$$

Proposition 2. *Let $\mathscr{E}$ be a strongly local, regular Dirichlet form and let $Y \subset X$. Then the following statements are equivalent:*

a) $1_Y \in \mathscr{D}_{\text{loc}}(\mathscr{E})$.

b) *The set Y is invariant for $\mathscr{E}$.*

c) *$T_t(1_Y \cdot u) = 1_Y \cdot T_t u$ for all $t > 0$ and all $u \in L^2(X, m)$.*

d) *The function 1_Y is excessive for $(T_t)_{t>0}$.*

e) *$Y = \{u > 0\}$ for some excessive function u.*

f) *$Y = \{u > 0\}$ for some function $u \in \mathscr{D}_{\text{loc}}(\mathscr{E})$ with $d\Gamma(u, u) \equiv 0$ on X.*

Moreover, in any of the above assertions one may replace Y by $X \setminus Y$ (since Y is invariant if and only if $X \setminus Y$ is invariant).

Proof. We only sketch the arguments. For the equivalence of b) and c) we refer to [O1]. The implications c) ⇒ d) ⇒ e) and b) ⇒ a) ⇒ f) are obvious. The implication e) ⇒ c) follows from general potential theoretic principles (cf. [DM], XII.19) applied to the semigroup $(P_t)_{t>0}$ of sub-Markovian kernels associated with the $L^2(X, m)$-semigroup $(T_t)_{t>0}$. It remains to prove that f) implies e). To this end, let $u \in \mathcal{D}_{\text{loc}}(\mathcal{E})$ with $d\Gamma(u, u) \equiv 0$ on X. Then (due to the truncation property) also $u_+ \in \mathcal{D}_{\text{loc}}(\mathcal{E})$ and $d\Gamma(u_+, u_+) \equiv 0$ on X. Cauchy-Schwarz inequality implies $d\Gamma(u_+, v) \equiv 0$ on X for all $v \in \mathcal{D}_{\text{loc}}(\mathcal{E})$, that is, u_+ is a nonnegative solution of the equation $Lu_+ = 0$ on X. According to Lemma 3, u_+ is excessive. □

We also recall that $\mathcal{E}$ is called *irreducible* if for every invariant set $Y \subset X$ either $m(Y) = 0$ or $m(X \setminus Y) = 0$.

Remarks. a) In view of the above Proposition, it is easy to see that a strongly local, regular Dirichlet form is irreducible if and only if for all $u \in \mathcal{D}_{\text{loc}}(\mathcal{E})$

$$d\Gamma(u, u) \equiv 0 \quad \text{on} \quad X \Leftrightarrow u \text{ is constant on } X.$$

In order to see the latter property, let $u_\lambda = (u - \lambda)_+$ for $\lambda \in \mathbb{R}$. By the truncation property, $u_\lambda \in \mathcal{D}_{\text{loc}}(\mathcal{E})$ with $d\Gamma(u_\lambda, u_\lambda) \equiv 0$ on X. Hence, the set $\{u_\lambda > 0\} = \{u > \lambda\}$ is invariant for $\mathcal{E}$. Since $\mathcal{E}$ was assumed to be irreducible this means that either $m\{u > \lambda\} = 0$ or $m\{u \leqq \lambda\} = 0$ and since this holds true for all $\lambda \in \mathbb{R}$ the function u must be constant.

It is also well-known that $\mathcal{E}$ is irreducible if and only if every excessive function u is either $\equiv 0$ or > 0 on X.

b) If $\mathcal{E}$ is irreducible then X is connected. The converse is in general not true. However, it holds true if all (locally bounded) solutions of the equation $Lu = 0$ on X are continuous. This is, for instance, the case in Riemannian geometry where the irreducibility of the canonical Dirichlet form on X is equivalent to the connectedness of the manifold X.

c) If we do not assume that $\mathcal{E}$ is irreducible then the assertion of Lemma 1 (and 1′) is still true with $X(x) = \bigcup_{r>0} B_r(x)$ instead of X. The assertions of Theorem 1 and 2 are changed into the statement that the respective functions are constant on each subset $Y \subset X$ on which $\mathcal{E}$ is irreducible.

If we assume that X is connected instead of $\mathcal{E}$ being irreducible) t hen Assumption (A) still implies that ϱ is a metric, in particular, $X(x) = X$ for any $x \in X$. Under these assumptions, the first assertion of Theorem 3 is still valid, namely, condition (3.1) implies that $\mathcal{E}$ is recurrent. (Note, however, that in this case "recurrence" just means that every potential Gf with $f \geqq 0$ only admits the values 0 and ∞, cf. [F].)

Under the same assumptions, all assertions of Theorem 4 are still valid. (In order to see this, it suffices to consider the claim on each irreducible subset of X.)

The assertions of Theorem 5 and Corollary 2 are even valid without assuming connectedness of X.

Acknowledgement. I would like to thank Profs. M. Biroli and U. Mosco for preprints of their papers [BM2] and [M] and for stimulating discussions during stays at Milano and Rome. I am also grateful to Prof. M. Takeda as well as to the referee for valuable comments.

References

[BM1] *M. Biroli* and *U. Mosco*, Formes de Dirichlet et estimation structurelles dans les milieux discontinus, C.R. Acad. Sci. Paris **313** (1991), 593–598.

[BM2] *M. Biroli* and *U. Mosco*, A Saint-Venant Principle for Dirichlet forms on discontinuous media, Elenco Preprint, Rome 1992.

[B] *R. Brooks*, A relation between growth and the spectrum of the Laplacian, Math. Z. **178** (1981), 501–508.

[CY] *S.Y. Cheng* and *S.T. Yau*, Differential Equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. **28** (1975), 333–354.

[D1] *E.B. Davies*, Heat kernels and spectral theory, Cambridge University Press, 1989.

[D2] *E.B. Davies*, L^1 properties of second order elliptic operators, Bull. London Math. Soc. **17** (1985), 417–436.

[DM] *C. Dellacherie* and *P.A. Meyer*, Probabilities and potential, C, North Holland, Amsterdam 1988.

[F] *M. Fukushima*, Dirichlet forms and Markov processes, North Holland and Kodansha, Amsterdam 1980.

[G1] *A.A. Grigor'yan*, On stochastically complete manifolds, Soviet Math. Dokl. **34** (1987), 310–313.

[G2] *A.A. Grigor'yan*, Stochastically complete manifolds and summable harmonic functions, Math. USSR Izvestiya **33** (1989), 425–431.

[I] *K. Ichihara*, Explosion problem for symmetric diffusion processes, Trans. Amer. Math. Soc. **298** (1986), 515–536.

[K1] *L. Karp*, Subharmonic functions, harmonic mappings and isometric immersions, in: Seminar on Differential Geometry (ed. by S.T. Yau), Ann. Math. Stud. **102**, Princeton 1982.

[K2] *L. Karp*, Subharmonic functions on real and complex manifolds, Math. Z. **179** (1982), 535–554.

[KL] *L. Karp* and *P. Li*, The heat equation on complete Riemannian manifolds, (unpublished) Preprint, Stanford and Ann Arbor 1983.

[Le] *Y. Lejan*, Mesures associées a une forme de Dirichlet. Applications, Bull. Soc. Math. France **106** (1978), 61–112.

[Li] *P. Li*, Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds, J. Diff. Geom. **20** (1984), 447–457.

[LS] *P. Li* and *R. Schoen*, L^p and mean value properties of subharmonic functions on Riemannian manifolds, Acta Math. **153** (1984), 279–301.

[L1] *T. Lyons*, Instability of the Liouville property for quasi-isometric Riemannian manifolds and reversible Markov chains, J. Diff. Geom. **19** (1987), 33–66.

[L2] *T. Lyons*, Instability of the conservative property under quasi-isometry, J. Diff. Geom. **34** (1991), 483–489.

[M] *U. Mosco*, Composite Media and asymptotic Dirichlet forms, Elenco Preprint, Rome 1992.

[O1] *Y. Oshima*, Lectures on Dirichlet forms, Lecture Notes, Erlangen 1988.

[O2] *Y. Oshima*, On conservativeness and recurrence criteria for Markov processes, Potential Analysis **1** (1992), 115–131.

[Sa] *L. Saloff-Coste*, Uniformly elliptic operators on Riemannian manifolds, J. Diff. Geom. **36** (1992), 417–450.

[St] *R.S. Strichartz*, Sub-Riemannian geometry, J. Diff. Geom. **24** (1986), 221–263.

[S1] *K.-Th. Sturm*, Analysis on local Dirichlet spaces – II. Upper Gaussian estimates for the fundamental solution of parabolic equations, Osaka J. Math., to appear.

[S2] *K.-Th. Sturm*, Analysis on local Dirichlet spaces – III. Poincaré and parabolic Harnack inequality, Preprint, Erlangen 1993.

[S3] *K.-Th. Sturm*, On the geometry defined by Dirichlet forms, in: Seminar on Stochastic Analysis, Random Fields and Applications (ed. by E. Bolthausen), Birkhäuser, to appear.

[Su] *D. Sullivan*, Related aspects of positivity in Riemannian geometry, J. Diff. Geom. **25** (1987), 327–351.

[T] *M. Takeda*, On a martingale method for symmetric diffusion processes and its applications, Osaka J. Math. **26** (1989), 605–623.

[V] *N.Th. Varopoulos*, The Poisson kernel on positively curved manifolds, J. Funct. Anal. **44** (1981), 359–380.

[Y1] *S.T. Yau*, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. **28** (1975), 201–228.

[Y2] *S.T. Yau*, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J. **25** (1976), 659–670.

Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstraße $1^1/_2$, D-91054 Erlangen

e-mail: sturm@mi.uni-erlangen.de

Eingegangen 1. März 1993, in revidierter Fassung 28. März 1994