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Stochastic Analysis

Exercise sheet 9 from 19/12/2008

Exercise 1 - Martingale representation (10 points)

Let B be a Brownian motion and $(\mathcal{F}_t^B)$ the (augmented) Brownian filtration. Define $\beta_t := \int_0^t \operatorname{sgn}(B_s)dB_s$.

i) Show that β is also an $(\mathcal{F}_t^B)$-Brownian motion.

ii) From local time theory one knows that $\beta_t = |B_t| + L_t$ where L denotes an $(\mathcal{F}_t^{|B|})$-adapted process (local time in 0). Deduce from this property that B has no representation of the form

$$B_t = c + \int_0^t H_s d\beta_s.$$

Exercise 2 - Time change (10 points)

i) Let M be a local martingale with $\langle M\rangle_\infty = \infty$ almost sure. Show that we have the following equation in the almost-sure-sense:

$$\limsup_{t\to 0} \frac{M_t}{\sqrt{2\langle M\rangle_t \log\log\langle M\rangle_t}} = 1.$$

ii) Prove the following converse to exercise 4.3: If M is a continuous local martingale vanishing at zero and if $\langle M\rangle$ is deterministic, then M is a Gaussian martingale and has independent increments.

Exercise 3 - Time discrete mass transformation (10 points)

i) The ordinary random walk with drift: Consider on the space $\{-1,1\}^{\mathbb{N}}$ the product measures $\mathbb{P}$ and $\mathbb{Q}$ that are uniquely defined by $\mathbb{P}(\omega_i = 1) = p \in (0,1)$ and $\mathbb{Q}(\omega_i = 1) = q \in (0,1)$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ denote the σ-Algebra generated by the first n coordinates. Show that for every n the restriction $\mathbb{P}|_{\mathcal{F}_n}$ is absolutely continuous with respect to $\mathbb{Q}|_{\mathcal{F}_n}$ and calculate the Radon-Nikodym density. Show that $\mathbb{P}$ is not absolutely continuous with respect to $\mathbb{Q}$ for $p \neq q$.

ii) Let $N_1, \ldots, N_n$ be independent $\mathcal{N}(0,1)$-distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mu \in \mathbb{R}^n$ a constant. Define a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$d\mathbb{Q}(\omega) := \exp\left(\sum_{i=1}^n \mu_i N_i(\omega) - \frac{1}{2}\sum_{i=1}^n \mu_i^2\right) d\mathbb{P}(\omega)$$

What is the distribution of the N_i under $\mathbb{Q}$?

Exercise 4 - An exponential martingale that is not a martingale (10 points)

Let B be a one-dimensional standard Brownian motion.

i) Show that the stopping time $T := \inf\{t : B_t^2 = 1 - t\}$ is almost surely finite.

ii) Define $H_s := -\frac{2}{(1-s)^2} B_s 1_{\{T\geq s\}}$. Show that for every $t \in [0,1]$ we have almost surely

$$\int_0^t H_s^2 ds < \infty.$$

iii) Define $M_t := \int_0^t H_s dB_s$. Show that $M_1 - \frac{1}{2}\langle M\rangle_1 \leq -1$ almost sure.

iv) Show that $\mathcal{E}(M)_1 < 1$ and therefore $\mathcal{E}(M)$ is not a martingale.