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# Stochastic Analysis

## Exercise sheet 8 from 12/12/2008

**Exercise 1 - Rotations of Brownian motion** (10 points)

Let $B$ be a $d$-dimensional Brownian motion and $O(t) = O_{ij}(t)$ a progressively measurable process taking its values in the set of $d \times d$-orthogonal matrices. Prove that the process $X$ defined by

$$X_t^i := \sum_{j=1}^{d} \int_0^t O_{ij}(s) dB^j(s)$$

is again a $d$-dimensional Brownian motion.

**Exercise 2 - Lévy's characterization theorem** (10 points)

Let $B = (B^{(1)}, B^{(2)}, B^{(3)})$ be a three-dimensional standard Brownian motion and define

$$X := \prod_{i=1}^{3} \operatorname{sgn}\left(B_1^{(i)}\right)$$

as well as the new coordinates

$$M_t^{(1)} := B_t^{(1)}, \quad M_t^{(2)} := B_t^{(2)}, \quad M_t^{(3)} := X B_t^{(3)}.$$

Show that each of the pairs $(M^{(1)}, M^{(2)})$, $(M^{(1)}, M^{(3)})$ and $(M^{(2)}, M^{(3)})$ is a two-dimensional Brownian motion, but $(M^{(1)}, M^{(2)}, M^{(3)})$ is *not* a three-dimensional Brownian motion. Explain why this does not provide a counterexample to Lévy's characterization theorem.

*Hint*: In the two-dimensional case show that the components constitute independent one-dimensional Brownian motions.

**Exercise 3 - Characterization of local martingales via exponential martingales** (10 points)

Let $M$ an $A$ be adapted continuous processes where in addition $A$ is of finite variation. Suppose, for every $\lambda \in \mathbb{R}$, the process

$$\exp\left(\lambda M_t - \frac{\lambda^2}{2} A_t\right)$$

is a local martingale. Show that $M$ is a local martingale with $\langle M \rangle = A$.

**Exercise 4 - Computation of exponential martingales by multiple Itô integrals** (10 points)

For $n \in \mathbb{N}_0$ define polynomials $H_n(x, y)$ by

$$H_n(x, y) := \frac{\partial^n}{\partial \alpha^n} \exp\left(\alpha x - \frac{1}{2}\alpha^2 y\right)\Big|_{\alpha=0}, \quad x, y \in \mathbb{R}.$$

i) Show that these polynomials satisfy the recursive relations

$$\frac{\partial}{\partial x} H_n(x, y) = n H_{n-1}(x, y)$$

for all $n \in \mathbb{N}$ as well as the backward heat equation

$$\frac{\partial}{\partial y} H_n(x, y) + \frac{1}{2} \frac{\partial^2}{\partial x^2} = 0$$

for all $n \in \mathbb{N}_0$.

ii) For any continuous local martingale $M$ verify the multiple Itô integral computation

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dM_{t_n} \ldots dM_{t_2} dM_{t_1} = \frac{1}{n!} H_n(M_t, \langle M \rangle_t)$$

and the expansion

$$\exp\left(\alpha M_t - \frac{\alpha^2}{2} \langle M \rangle_t\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(M_t, \langle M \rangle_t).$$