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Stochastic Analysis

Exercise sheet 7 from 11/28/2008

Exercise 1 - Transience of the Brownian motion (10 points)

Let B be a two-dimensional Brownian motion that starts at $x \neq 0$. Define the stopping times

$$T_k := \inf\Big\{t \geq 0 : |B_t| \notin \big](1/k)^k, k\big[\Big\}.$$

i) Chose $k \in \mathbb{N}$ such that $\left(\frac{1}{k}\right)^k < |x| < k$. Use Itô's formula to show that $\log |B^{T_k}|$ is a bounded martingale.

ii) Compute the probability $\mathbb{P}\left[|B_{T_k}| = \left(\frac{1}{k}\right)^k\right]$ for such k and argue that B will never hit the origin of ordinates with probability one. Generalize this result to higher dimensions.

Exercise 2 - Exit times (10 points)

Let B be a d-dimensional Brownian motion that starts at $x \neq 0$. For $a > 0$ define the stopping time $T_a := \inf\{t : |B_t| = a\}$.

i) Let $d = 2$. Show that $\log |B|$ is a local martingale. For $0 < r < |x| < R$ prove the identity

$$\mathbb{P}(T_r < T_R) = \frac{\log R - \log |x|}{\log R - \log r}.$$

What about $\mathbb{P}(T_r < \infty)$?

ii) Let $d \geq 3$. Show that $|B|^{2-d}$ is a local martingale. For $0 < r < |x| < R$ prove the identity

$$\mathbb{P}(T_r < T_R) = \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}}.$$

Show that this implies $\mathbb{P}(T_r < \infty) = \frac{r^{d-2}}{|x|^{d-2}}$.

Exercise 3 - A uniformly integrable proper local martingale (10 points)

Let B be a three-dimensional Brownian motion that starts at $x \neq 0$. Show

i) $X := \frac{1}{|B|}$ is a local martingale.

ii) There is no integrable random variable that dominates X.

iii) X is uniformly integrable. For this purpose you may show for all $p > 0$ the inequality

$$\sup_{t \geq 0} \mathbb{E}\left[X_t^p\right] \leq c \int_{B_1^3(0)} \frac{1}{|y|^p} dy + 1$$

where $B_1^3(0) := \{z \in \mathbb{R}^3 : |z| < 1\}$.

iv) X is not a martingale. For this purpose you may show for all $p > 0$ and $t \geq 0$ the inequality

$$\mathbb{E}[X_t^p] \leq \frac{1}{2t^{\frac{p}{2}}} \int_0^\infty y^{-\frac{p}{2}} e^{-\frac{1}{2}y} dy.$$

v) $\lim_{t \to \infty} |B_t| = \infty$ almost sure.

Hint: Use exercise 2ii) to solve part ii).

Remark: One can show that a Brownian motion (or more general a Lévy process) is either recurrent or transient. Consequently the results of exercises 1ii) and 3iv) are actually equivalent. As we have to use exercise 1 to define well the process X the last result should be considered as a by-product of the calculations above.

Exercise 4 - Exponential inequality (10 points)

Let M be a continuous local martingale vanishing at 0. Suppose there is a constant $c > 0$ such that $\langle M \rangle_t \leq ct$ for all $t \geq 0$. Use the maximal inequality to show that for all $a > 0$

$$\mathbb{P}\left(\sup_{s \leq t} M_s \geq at\right) \leq \exp\left(-\frac{a^2 t}{2c}\right).$$

The case $c = 1$ yields the Brownian case.