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Stochastic Analysis

Exercise sheet 6 from 11/21/2008

Exercise 1 - Examples of Lebesgue-Stieltjes integrals (10 points)

Calculate explicitly the following Stieltjes integrals:

$$i)\quad \int_0^T (\sin(x)d\cos(x) + \cos(x)d\sin(x)).$$
$$ii)\quad \int_0^T 1_{[1,y)}(x)d|x-2|.$$
$$iii)\quad \int_0^T |x-2|d_{[1,y)}(x).$$

Exercise 2 - Simple integration by parts (10 points)

Let f and g be two real functions with finite variation on compacts. Show that the following integration-by-parts formula then holds for all $t \geq 0$:

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s)dg(s) + \int_0^t g(s-)df(s).$$

As usual $f(s-) := \lim_{u\uparrow s} f(u)$ denotes the left limit of the function f at point s.

Exercise 3 - Linearity of the Itô-integral operator (10 points)

Let X be a continuous semimartingale and let $\mathcal{B}^0$ denote the algebra of all bounded $\mathcal{F}_0$-measurable random variables. Show that the map

$$\mathcal{B} \to \mathcal{S}, \quad H \mapsto H \bullet X$$

ist $\mathcal{B}_0$-linear, that is to say: For all $a, b \in \mathcal{B}_0$ and $H, K \in \mathcal{B}$ we have

$$\int_0^\cdot (aH + bK)dX = a\int_0^\cdot HdX + b\int_0^\cdot KdX.$$

As usual $\mathcal{B}$ denotes the set of all adapted, left-continuous processes that are pathwise locally bounded and $\mathcal{S}$ is the set of all continuous semimartingales.

Exercise 4 - Stochastic integrals and derivatives (10 points)

Let H be an adapted continuous process and B a Brownian motion. The aim of this exercise is the calculation of the following stochastic derivation

$$\lim_{h\to 0} \frac{1}{B_{t+h} - B_t} \int_t^{t+h} H_s dB_s$$

for every $t \geq 0$, where the limit is taken in probability. As we will see, the result will be similar to the Fundamental Theorem of Calculus.

Fix $t \geq 0$ and let $\epsilon > 0$.

i) Show that there exists $K_\epsilon > 0$, independent of t, such that

$$\mathbb{P}\left(\left|\frac{1}{B_{t+h} - B_t}\int_t^{t+h}(H_s - H_t)dB_s\right| > \epsilon\right) \leq \mathbb{P}\left(\left|\frac{1}{\sqrt{h}}\int_t^{t+h}(H_s - H_t)dB_s\right| > \frac{\epsilon}{K_\epsilon}\right) + \frac{\epsilon}{2}.$$

ii) Define the stopping time

$$T_\delta := \inf\{s \in [t, t+\delta] : |H_s - H_t|^2 \geq \frac{\epsilon^3}{4K_\epsilon^2}\}.$$

Show that for every $h \leq \delta$ we have

$$\mathbb{P}(|\frac{1}{\sqrt{h}} \int_t^{t+h} (H_s^T - H_t)dB_s| > \frac{\epsilon}{K_\epsilon}) \leq \frac{\epsilon}{4}.$$

iii) Show that there exists $\delta > 0$ such that for all $h \leq \delta$ we get $\mathbb{P}(H^T \neq H$ on the set $[t, t+h]) \leq \frac{\epsilon}{4}$ and consequently

$$\mathbb{P}(|\frac{1}{B_{t+h} - B_t} \int_t^{t+h} (H_s - H_t)dB_s| > \epsilon) \leq \epsilon.$$

The amount of the previous results gives rise to

$$\mathbb{P} - \lim_{h \to 0} \frac{1}{B_{t+h} - B_t} \int_t^{t+h} H_s dB_s = H_t.$$