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Stochastic Analysis

Exercise sheet 5 from 11/14/2008

Exercise 1 - Lebesgue-Stieltjes integral (10 points)

i) Let $f \in \mathcal{C}^1(\mathbb{R})$ and let g be a function of finite variation. Show that the process $f' \circ g$ is integrable with respect to the measure dg and that the following formula holds

$$f(g(t)) - f(g(0)) = \int_0^t f'(g(s))dg(s).$$

Recall that dg denotes the (signed) Radon measure, associated with g.

ii) Let $g \in \mathcal{C}^1(\mathbb{R}_+)$ be a smooth integrator. Show that in this case the Lebesgue-Stieltjes integral turns out to be an ordinary Lebesgue integral with density g'. In other words: For every locally bounded and Borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ we have

$$\int_0^t f(s)dg(s) = \int_0^t f(s)g'(s)ds.$$

Exercise 2 - Naive stochastic integration (10 points)

Let g be a right-continuous function and let Δ_n be a refining partition of $[0, 1]$ with $\lim_{n\to\infty} \|\Delta_n\| = 0$. For any continuous function $f \in \mathcal{C}^0([0, 1])$ define the sums

$$S_n^f := \sum_{t_k, t_{k+1} \in \Delta_n} f(t_k)(g(t_{k+1}) - g(t_k)).$$

Assume now that for all $n \in \mathbb{N}$ and for all $h \in \mathcal{C}^0([0, 1])$ the limit $\lim_{n\to\infty} S_n^f$ exists and is finite. Show that this forces g to be of finite variation.

Hint: Banach-Steinhaus Theorem: Let X be a Banach space and let Y be a normed linear space. Let $\{T_\alpha\}$ be a family of continuous linear operators from X into Y. If for each $x \in X$ the set $\{T_\alpha x\}$ is bounded, then the set $\{T_\alpha\}$ is bounded. Chose $X := (\mathcal{C}^0([0, 1]), \|.\|_{\sup})$ and $Y := (\mathbb{R}, |.|)$.

Remark: As a consequence of this result, a definition of the stochastic integral for continuous processes as the limit of Riemann sums will fail in cases where we chose the integrator to be a local martingale. Nevertheless, in order to define the stochastic integral intuitively one has to restrict oneself to predictable processes.

Exercise 3 - Calculation of a stochastic integral (10 points)

Let B be a standard Brownian motion. Let $\epsilon \in [0, 1]$ and let $\Delta = \{t_0, t_1, \ldots, t_m\}$ be a partition of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_m = t$. Define

$$S_\epsilon(\Delta) := \sum_{i=0}^{m-1} ((1 - \epsilon)B_{t_i} + \epsilon B_{t_{i+1}})(B_{t_{i+1}} - B_{t_i}).$$

Show that in the L^2-sense we have

$$\lim_{\|\Delta\|\to 0} S_\epsilon(\Delta) = \frac{1}{2}B_t^2 + (\epsilon - \frac{1}{2})t$$

In particular, for $\epsilon = 0$, this yields: $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$. By using the result from exercise 1i) and the path-by-path definition of the stochastic integral for processes of finite variation compare this to the integral $\int_0^t A_s dA_s$ for any $A \in \mathcal{A}$.

Remark: The above comparison motivates the definition of the so called Stratonovich integral: $\int_0^t Y_s \circ dX_s := \int_0^t Y_s dX_s + \frac{1}{2}\langle X, Y\rangle$. Indeed one can show that in this notation one regains an ordinary integration by parts formula.

Exercise 4 - Deterministic integrands (10 points)

Let B be a Brownian motion.

i) Let f be a left-continuous function on $[0, 1]$. Prove that the random variable

$$Z := \int_0^1 f(s) dB_s$$

is Gaussian and compute its variance.

ii) For $m \geq 1$ define the stochastic integrals

$$A_m := \sqrt{2} \int_0^1 \cos(2\pi m t) dB_t, \; B_m := \sqrt{2} \int_0^1 \sin(2\pi m t) dB_t.$$

Show:

(a) $\forall m \geq 1 : A_m, B_m \sim \mathcal{N}(0, 1)$.

(b) $\forall m \geq 1 : A_m$ and B_m are not correlated.

Hint: Prove $i)$ for step functions first. Then argue that L^2-limits are again normally distributed. For $ii)(b)$ use part $i)$ and polarization.