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Stochastic Analysis

Exercise sheet 4 from 11/07/2008

Exercise 1 - Intervals of constancy (10 points)

Let M be a continuous local martingale. Give a proof of:

i) For all $a < b$ and almost every ω holds:

$$\langle M\rangle_a(\omega) = \langle M\rangle_b(\omega) \iff \forall t \in [a,b] : M_t(\omega) = M_a(\omega).$$

ii) Let $\mathbb{P}\left(\sup_{t\geq 0}\langle M\rangle_t < \infty\right) = 1$. Then $(M_t)_t$ converges almost surely.

Exercise 2 - Weak independency (10 points)

i) Let M, N be two independent continuous local martingales. Show that $\langle M, N\rangle = 0$. In particular, if B is a d-dimensional Brownian motion, prove that its components fulfil $\langle B^i, B^j\rangle_t = \delta^{ij} t$.

ii) Prove that the converse to the reult in *i)* is false. For this purpose consider the processes X^T and Y^T where (X,Y) denotes a two-dimensional Brownian motion and T a stopping time that is not constant.

Exercise 3 - Characterization of Gaussian processes (10 points)

Let M be a continuous local martingale and a Gaussian process. Prove that $\langle M\rangle$ is deterministic, i.e. there is a function f on $\mathbb{R}_+$ such that $\langle M\rangle_t = f(t)$ a.s.

Remark: As the converse to this result is also true, one can regard it as a generalization of Lévy's characterization theorem.

Exercise 4 - P-th variation (10 points)

Let $\Pi := \{t_0, t_1, \ldots, t_m\}$, $0 = t_0 \leq t_1 \ldots \leq t_m = t$, be a partitition of the interval $[0,t]$ and $\|\Pi\| := \max_{1\leq k\leq m}|t_k - t_{k+1}|$ the fineness of the partitition. For $p > 0$ define the p-th variation of a process X with respect to the partitition Π by

$$V_t^{(p)}(\Pi) := \sum_{k=1}^{m} |X_{t_k} - X_{t_{k-1}}|^p.$$

Let $p > 0$ and let X be a continuous adapted process for which holds

$$\lim_{\|\Pi\|\to 0} V_t^{(p)}(\Pi) = L_t$$

stochastically for all $t > 0$, where L_t denotes for every $t > 0$ a random variable that takes its values in the set $[0,\infty]$ almost surely. Prove:

i) $\lim_{\|\Pi\|\to 0} V_t^{(q)}(\Pi) = \infty$ on the set $\{L_t > 0\}$ for $0 < q < p$.

ii) $\lim_{\|\Pi\|\to 0} V_t^{(q)}(\Pi) = 0$ on the set $\{L_t < \infty\}$ for $q > p$.

What does this imply for the p-th variation of a Brownian motion?