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Stochastic Analysis

Exercise sheet 3 from 10/31/2008

Exercise 1 – Characterization of the Brownian motion (10 points)

Let B denote a one-dimensional continuous process, starting at 0. Suppose that for all $\alpha \in \mathbb{R}$ the process M, defined by

$$M_t := \exp\left(\alpha B_t - \frac{\alpha^2}{2} t\right)$$

for all $t \in \mathbb{R}_+$, is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that B is an $(\mathcal{F}_t)$-Brownian motion, i.e.

- The distribution of $B_t - B_s$ equals $\mathcal{N}(0, t-s)$.
- $B_t - B_s$ is independent from $\mathcal{F}_s$.

Since we already know that the converse is also true, this gives a characterization of the Brownian motion.

Hint: The Laplace transform of a random variable X characterizes uniquely its distribution. In particular, X is $\mathcal{N}(0,1)$-distributed iff $\mathbb{E}\left[e^{\lambda X}\right] = e^{\frac{\lambda^2}{2}}$ for all $\lambda \in \mathbb{R}$.

Exercise 2 - Construction of a random walk (10 points)

Let $(M_t)_{t\geq 0}$ be a continuous martingale on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that

$$\mathbb{P}\left(\sup_{t\geq 0} M_t = +\infty, \inf_{t\geq 0} M_t = -\infty\right) = 1.$$

Define $T(0) := 0$ and $T(n) := \inf\{t > T(n-1) : |M_t - M_{T(n-1)}| = 1\}$ for $n \geq 1$. Show that the discrete process $\left(M_{T(n)}\right)_{n\in\mathbb{N}}$ constitutes an ordinary random walk, i.e. there exists an iid-sequence $(\xi_i)_{i\in\mathbb{N}}$, where ξ_1 is uniformly distributed on the set $\{-1, 1\}$, such that for all $n \in \mathbb{N}$

$$M_{T(n)} = \sum_{i=1}^{n} \xi_i.$$

Exercise 3 - Poisson process (10 points)

Let $(N_t)_{t\geq 0}$ be a càdlàg version of an $(\mathcal{F}_t)$-Poisson process with intensity $\lambda > 0$, starting at 0. Recall that this means exactly:

- The distribution of $N_t - N_s$ equals $\mathcal{P}(\lambda(t-s))$.
- $N_t - N_s$ is independent from $\mathcal{F}_s$.

Furthermore define the process X by

$$X_t := \exp\left(iuN_t - \lambda t(e^{iu} - 1)\right)$$

for $u \in \mathbb{C}$ and for all $t \in \mathbb{R}_+$. Show:

i) The compensated Poisson process $(N_t - \lambda t)_{t\geq 0}$ is a martingale.
ii) The process $((N_t - \lambda t)^2 - \lambda t)_{t\geq 0}$ is a martingale.
iii) The processes $(\Re(X_t))_{t\geq 0}$ and $(\Im(X_t))_{t\geq 0}$ are martingales.
iv) Consider X with $u = -i$. Does this martingale converge in L^1?

Exercise 4 - Quadratic variation (10 points)

Let X be a centred, continuous and square integrable process that has stationary and independent increments. Prove that we have in this case:

$$\langle X \rangle_t = t\mathbb{E}[X_1^2]$$

for all $t \geq 0$, where the underlying filtration is supposed to be generated by X.