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Stochastic Analysis

2. Exercise sheet from 10/24/2008

Exercise 1 – Backward martingales (10 points)

Let $(X_n, \mathcal{F}_n)_{n\in -\mathbb{N}}$ be a so called backward submartingale, that is for all $n \le m \le 0$ we have $\mathcal{F}_n \subset \mathcal{F}_m$ (decreasing sequence of sub-σ-fields) and $X_n \le \mathbb{E}[X_m|\mathcal{F}_n]$. Prove:

i) $\lim_{n\to-\infty} X_n$ exists a.s.

ii) Let $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|] < \infty$. Then the family $\{X_n : n \in -\mathbb{N}\}$ is uniformly integrable. In particular $(X_n)_{n\in-\mathbb{N}}$ converges in L^1. Moreover, for every $n \in -\mathbb{N}$ the inequality

$$\lim_{k\to-\infty} X_k \le \mathbb{E}[X_n|\mathcal{F}_{-\infty}]$$

holds, where $\mathcal{F}_{-\infty} := \bigcap_{n\in-\mathbb{N}} \mathcal{F}_n$.

Exercise 2 - Convergence theorem (10 points)

Let $(X_t, \mathcal{F}_t)_{t\in\mathbb{R}_+}$ be a nonnegative, right-continuous submartingale. Then the following conditions are equivalent:

i) The family $\{X_t : t \in \mathbb{R}_+\}$ is uniformly integrable.

ii) $\lim_{t\to\infty} X_t$ exists in L^1.

iii) There exists an integrable random variable X_∞ such that

- $\lim_{t\to\infty} X_t = X_\infty$ a.s.
- $(X_t, \mathcal{F}_t)_{t\in\mathbb{R}_+\cup\{+\infty\}}$ is a submartingale, where we define $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t\right)$.

Exercise 3 - Exit times in one dimension (10 points)

Let $(B_t)_{t\in\mathbb{R}_+}$ be a one-dimensional standard Brownian motion. For $a, b > 0$ and $x \in \mathbb{R}$ define the stopping times

$$T_x := \inf\{t \in \mathbb{R}_+ : B_t = x\}$$
$$T_{a,b} := \inf\{t \in \mathbb{R}_+ : B_t = -a \vee B_t = b\} = T_{-a} \wedge T_b.$$

Consider the process $X_t := \sinh(\theta(B_t + a)) \exp(-\frac{\theta^2}{2}t)$ for $t \in \mathbb{R}_+$.

i) Show that $(X_t)_{t\in\mathbb{R}_+}$ is a martingale.

ii) For all $\lambda \ge 0$ we have

$$\mathbb{E}[\exp(-\lambda T_{a,b})] = \frac{\cosh(\frac{a-b}{2}\sqrt{2\lambda})}{\cosh(\frac{a+b}{2}\sqrt{2\lambda})}.$$

iii) Calculate the distribution of $\sup_{0\le t\le T_{-1}} B_t$.

Hint: In ii) apply i) and optional sampling. Part iii) is independent of the first two results.

Exercise 4 - Exit times in n dimensions (10 points)

Let $(B_t)_{t\in\mathbb{R}_+}$ be an n-dimensional standard Brownian motion.

i) Show that for every stopping time T with $\mathbb{E}[T] < \infty$ one has:

a) $\mathbb{E}[B_T] = 0$.

b) $\mathbb{E}\left[|B_T|^2\right] = n\mathbb{E}[T]$.

ii) For $r > 0$ and $x \in \mathbb{R}^n$ define the exit time $T_{r,x} := \inf\{t \in \mathbb{R}_+ : |B_t - x| \ge r\}$. Show that

$$\mathbb{E}[T_{r,x}] = \begin{cases} \frac{r^2-|x|^2}{n} & , |x| < r \\ 0 & , \text{otherwise} \end{cases}.$$