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Stochastic Analysis

Exercise sheet 12 from 01/23/2009

Exercise 1 - Khas'minskii lemma (10 points)
Let B denote a d-dimensional Brownian motion and $V : \mathbb{R}^d \to \mathbb{R}_+$ a measurable function such that

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\int_0^t V(X_s)ds \right] < 1. \tag{1}$$

where x denotes the starting point of the Brownian motion.

i) Show that property (1) is sufficient for

$$\forall t \geq 0, \forall x \in \mathbb{R}^d : \quad \mathbb{E}_x \left[\exp\left(\int_0^t V(X_s)ds \right) \right] < \infty.$$

ii) Suppose there exist constants $C_1, C_2 \in \mathbb{R}_+$ and an index $\alpha \in (0,1)$ such that

$$0 \leq V(x) \leq C_1 + C_2 \frac{1}{\|x\|^{2\alpha}}.$$

Show that this implies (1).

Exercise 2 - Scale function I (10 points)
For $-\infty \leq l < r \leq +\infty$ consider on $]l,r[$ the one-dimensional stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \in]l,r[\tag{2}$$

which is supposed to submit a unique solution, denoted again by X. Furthermore let the coefficients b and σ as well as $\frac{1}{\sigma^2}$ be measurable and locally bounded on $]l,r[$. A strictly monotonically increasing function $s :]l,r[\to \mathbb{R}$ is called a *scale function* of X if the process $s(X) - s(X_0)$ is a local martingale.

i) Consider the ordinary differential equation

$$\frac{1}{2}\sigma^2 s'' + bs' = 0 \tag{3}$$

on $]l,r[$. Show that every non-constant solution s of (3) is either strictly monotonically increasing or decreasing. Prove that s or $-s$ is a scale function of X.

ii) Let $s \in \mathcal{C}^2(]l,r[)$ be a scale function of X. Show that s solves the equation (3).

iii) Show that for every $x_0 \in]l,r[$ the function

$$s(y) := \int_{x_0}^y \exp\left(-2 \int_{x_0}^t \frac{b}{\sigma^2}(s)ds \right) dt, \quad y \in]l,r[$$

is a scale function of X.

Exercise 3 - Scale function II (10 points)
Let $s \in \mathcal{C}^2(]l,r[)$ be a scale function for the solution X of the equation (2).

i) Show that $Y := s(X)$ solves the equation

$$dY_t = \widetilde{\sigma}(Y)dB_t$$

where $\widetilde{\sigma} := (\sigma s') \circ s^{-1}$.

ii) For $l < y < x < z < r$ show that

$$\mathbb{P}_x \left[T_y < T_z\right] = \frac{s(z) - s(x)}{s(z) - s(y)}.$$

As usual $T_y := \inf\{t \geq 0 : X_t = y\}$ denotes the hitting time of the point set $\{y\}$.

Exercise 4 - Square of δ-dimensional Bessel process (BESQ$^\delta$) (10 points)

Let $\delta \geq 0$ be a nonnegative real number. Consider the one-dimensional stochastic differential equation

$$dZ_t = \delta dt + 2\sqrt{|Z_t|}dB_t, \quad Z_0 = x \geq 0. \tag{4}$$

One can show by uniqueness and existence results for SDE's with Hölder coefficients that this equation submits a unique strong solution (*square of δ-dimensional Bessel process*, abbreviated by BESQ$^\delta$). Denote it by Z.

i) Show that $Z_t \geq 0$ almost sure for every $t \geq 0$. In particular we can omit the modulus in equation (4).

ii) Define on the positive axis $]0, \infty[$ the functions

$$s_\nu(x) := x^{-\nu} \text{ for } \nu < 0, \qquad s_\nu(x) := -x^{-\nu} \text{ for } \nu > 0.$$

Show that s_ν is a scale function for the BESQ$^\delta$ with $\nu = \frac{\delta}{2} - 1$. Use then exercise 3*ii*) to argue that

$$\tau_0 := \inf\{t \geq 0 : Z_t = 0\} \begin{cases} < +\infty & \text{a.s. for } \nu < 0 \\ = +\infty & \text{a.s. for } \nu > 0. \end{cases}$$

iii) Define $X_t := \sqrt{Z_t}$. Show that for $\delta > 2$ the process X is a strong solution of the Bessel SDE

$$dX_t = \frac{\delta - 1}{2X_t}dt + B_t, \quad X_0 = \sqrt{x}.$$

We call X Bessel process of dimension δ.