Institut für angewandte Mathematik
Universität Bonn
Prof. Dr. K.-Th. Sturm
Frank Miebach
http://www-wt.iam.uni-bonn.de/~sturm/vorlesungWS0809/

Stochastic Analysis

Exercise sheet 11 from 01/16/2009

Exercise 1 - Girsanov theorem (10 points)

Let X denote the solution of the one-dimensional stochastic differential equation

$$dX_t = \mu dt + \sigma dB_t, \quad X_0 = x$$

with constants $\mu, \sigma \in \mathbb{R}$. Let h be a $\mathcal{C}^2$-function such that $h(X_t)$ is a positive martingale. Define a consistent family of measures by

$$d\mathbb{Q}|_{\mathcal{F}_t} := \frac{h(X_t)}{h(x)} d\mathbb{P}|_{\mathcal{F}_t}$$

and consider a $\mathbb{P}$-martingale M. Show that

$$M_t - \int_0^t \frac{h'(X_s)}{h(X_s)} d\langle M, X\rangle_s$$

is a $\mathbb{Q}$-martingale.

Exercise 2 - Novikov condition (10 points)

The aim of this exercise is to deduce a sufficient condition for an exponential martingale to be a martingale. For this purpose let M be a continuous local martingale and define

$$Z_t := \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$$

for $0 \le t < \infty$. Suppose that $\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M\rangle_t\right)\right] < \infty$ for all $t \ge 0$ (*Novikov condition*).

i) Define the process $N_t := \exp\left(B_{t\wedge S_n} - \frac{1}{2}(t \wedge S_n)\right)$ wherein $S_n := \inf\{s \ge 0 : B_s - s = -n\}$ and B is a Brownian motion. Show that $(N_t)_{0\le t<\infty}$ is a martingale. Use Fatou's lemma to a argue that the same holds for $(N_t)_{0\le t\le\infty}$. In particular the optional sampling theorem is applicable and one gets $\mathbb{E}\left[\exp\left(B_{R\wedge S_n} - \frac{1}{2}(R \wedge S_n)\right)\right] = 1$ for any stopping time R.

ii) Apply *i)* to $R := \langle M\rangle_t$ for fixed $t \ge 0$ and use the Dubins-Schwarz theorem as well as the obove hypotheses to show that $\mathbb{E}[Z_t] = 1$ for all $t \ge 0$. Therefore Z is a martingale.

<u>Remark</u>: The optional sampling theorem also works for (sub-)martingales with a last element. In this case the occurring stopping times don't have to be bounded.

Exercise 3 - A nonlinear stochastic differential equation (10 points)

Solve explicitly the one-dimensional stochastic differential equation

$$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}\,dB_t, \qquad X_0 = \zeta.$$

<u>Hint</u>: If you can't guess the right solution, it may help to try Doss' method (sheet 10, exercise 4).

Exercise 4 - Comparison principle (10 points)

Consider the two one-dimensional stochastic differential equations

$$X_t^1 = \zeta_1 + \int_0^t b_1(s, X_s^1)ds + \int_0^t \sigma(s, X_s^1)dW_s$$
$$X_t^2 = \zeta_2 + \int_0^t b_2(s, X_s^2)ds + \int_0^t \sigma(s, X_s^2)dW_s$$

with the same dispersion σ and the same Brownian motion W. Assume that all coefficients are Lipschitz and that there exist unique solutions. Suppose furthermore $\zeta_1 \leq \zeta_2$ almost sure and $b_1(s,x) \leq b_2(s,x)$ for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$. Show that $X_t^1 \leq X_t^2$ almost sure for all $t \geq 0$. In particular the solution of the equation

$$X_t = \lambda + \int_0^t b(X_s)ds + \int_0^t X_s dW_s$$

with nonnegative drift $b : \mathbb{R} \to \mathbb{R}_+$ and $\lambda \geq 0$ is itself nonnegative.

Hint: Show $\mathbb{E}\left[\left(X_t^1 - X_t^2\right)_+\right] = 0$ *for all* $t \geq 0$ *by applying Itô's formula to an appropriate approximation of* $\left(X_t^1 - X_t^2\right)_+$ *and then using Gronwall's lemma.*