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Stochastic Analysis

Exercise sheet 10 from 01/09/2009

Exercise 1 - Representation of Brownian functionals (10 points)

Let $(B_t)_{0\le t\le 1}$ be a one-dimensional Brownian motion and f a continuous function such that $f(B_1)$ is an L^2-random variable. Find a martingale representation for the Brownian martingale $(\mathbb{E}[f(B_1)|\mathcal{F}_t])_{0\le t\le 1}$ and thus a representation for the functional $f(B_1)$.

Hint: Use the Markov property of a Brownian motion and Itô's formula.

Exercise 2 - Brownian motion with drift (10 points)

Let B be a one-dimensional Brownian motion and let $a, b, \gamma \in \mathbb{R}$ be constants.

i) Define the stopping time $T_a := \inf\{t \ge 0 : B_t = a\}$. Show that

$$\mathbb{P}[T_a \le t] = 2\mathbb{P}[B_t \le a, T_a \le t] = 2\mathbb{P}[B_t \ge a] = \int_0^t \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}} ds.$$

ii) Let $X_t := B_t + \gamma t$ denote a Brownian motion with drift and consider the stopping time $\tau_b := \inf\{t \ge 0 : X_t = b\}$. Use Girsanov's theorem and part *i)* to calculate the distribution function $t \mapsto \mathbb{P}[\tau_b \le t]$ and the probability $\mathbb{P}[\tau_b < \infty]$.

iii) Compute the distribution of the random variable $\sup_{0\le s\le t} X_s$.

Exercise 3 - Linear stochastic differential equations (10 points)

Solve the one-dimensional linear stochastic differential equation

$$dX_t = rX_t dt + \sigma X_t dB_t$$

with some initial data X_0 and constants r and σ.

Exercise 4 - Doss' method (10 points)

Consider the one-dimensional stochastic differential equation

$$X_t = \zeta + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

with a two times differentiable dispersion σ that has bounded derivatives. Suppose that

$$\tilde{b}(x) := b(x) - \frac{1}{2}\sigma(x)\sigma'(x)$$

is Lipschitz. The following two steps will help to construct a solution to the above equation.

i) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) := \frac{\tilde{b}(u(x,y))}{\frac{\partial u}{\partial y}(x,y)}$ wherein u denotes the solution of the ordinary differential equation

$$\frac{\partial u}{\partial x}(x,y) = \sigma(u(x,y)), \quad u(0,y) = y.$$

Show that for all $y_0 \in \mathbb{R}$ and for all continuous functions $g : \mathbb{R}_+ \to \mathbb{R}$ the equation

$$\dot{y}(t) = f(g(t), y(t)), \quad y(0) = y_0$$

admits a unique solution $y : \mathbb{R}_+ \to \mathbb{R}$. Apply the result to $g(t) := W_t(\omega)$ and $y_0 := \zeta(\omega)$ for fixed ω and denote the corresponding solution by $Y_t(\omega)$.

ii) Define $X_t(\omega) := u(W_t(\omega), Y_t(\omega))$ and apply Itô's formula to show that X is a solution of the stochastic differential equation.

iii) Solve the equation from exercise 3 by means of Doss' method.