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Markov Processes

Exercise sheet 9 from 06/18/2009

Exercise 1: A set of cylindric-functions (10 points)

Let H be a separable Hilbert space, $\{W_h\}_{h\in H}$ a Gaussian process with measure space $(\Omega, \mathcal{P}, \mathbb{P})$. Consider the set

$$\mathcal{P} := \{p(W_{h_1}, \dots, W_{h_n}) | p \text{ polynomial}, \ h_i \in H \text{ for } i = 1, \dots, n\}$$

Show that $\mathcal{P}$ is dense in $L^r(\Omega)$ for all $r \geq 1$.
Hint: Consider a function $Z \in L^{\frac{r}{r-1}}(\Omega)$, $r > 1$. Show that if $\mathbb{E}[ZY] = 0$ for all $Y \in \mathcal{P}$, it follows that $Z = 0$.

Exercise 2 : A chaos decomposition (10 points)

i) Let $F \in \mathcal{C}^\infty(\mathbb{R})$ and $F^{(m)} \in L^2(\mathbb{R}, \nu)$ for all $m \in \mathbb{N}_0$, where $\nu := N(0,1)$. Show that

$$F = \sum_{m=0}^{\infty} c_m H_m, \text{ where } c_m := \int_{\mathbb{R}} F^{(m)}(x)\nu(dx).$$

ii) Now let $F \in \mathbb{D}^{\infty,2} := \bigcap_{k\in\mathbb{N}} \mathbb{D}^{k,2}$. Show that

$$F = \sum_{m=0}^{\infty} I_m(f_m), \text{ where } f_m := \frac{1}{m!}\mathbb{E}[D^m F].$$

Exercise 3: A chain rule (10 points)

Let $F = (F_1, \dots, F_m) \in (\mathbb{D}^{1,p})^m$, $p \geq 1$ and $\varphi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R})$ with bounded partial derivatives. Show that $\varphi(F) \in \mathbb{D}^{1,p}$ and find a "chain rule" to express $D(\varphi(F))$.

Exercise 4: Derivative of $\sup_{t\in[0,1]} W_t$ (10 points)

Let $\{W_t\}_{t\in[0,1]}$ be a standard one-dimensional Brownian motion and $M := \sup_{t\in[0,1]} W_t$. Show that $M \in \mathbb{D}^{1,2}$ and $D_t(M) = \chi_{[0,T]}(t)$, where T is the (a.s. unique) time for which the process W takes its maximum.