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Markov Processes

Exercise sheet 8 from 06/12/2009

Exercise 1: The differential operator on the space $L^2(\mathbb{R}, \mathcal{N}(0,1))$ (10 points)

Consider on the space $L^2(\mathbb{R}, \mathcal{N}(0,1))$ the operator $Du := u'$ and denote by H_n as usual the n-th Hermite polynomial.

i) Determine the adjoint operator D^* and show that the *Ornstein-Uhlenbeck operator* $A := -D^*D$ fulfills

$$(Au)(x) = u''(x) - xu'(x).$$

ii) Show for every $n \in \mathbb{N}_0$ the identities

$$DH_n = H_{n-1}, \qquad D^*H_n = (n+1)H_{n+1}, \qquad AH_n = nH_n.$$

Therefore we call D the *annihilation operator*, D^* the *generating operator* and A the *number operator*.

iii) Show that the functions $(\sqrt{n!}H_n)_{n\in\mathbb{N}_0}$ constitute a complete orthonormal system in $L^2(\mathbb{R}, \mathcal{N}(0,1))$.

iv) Show the commutator relation $DD^* - D^*D = \mathrm{Id}$.

Exercise 2: Multiple stochastic integrals (10 points)

Let $(T, \mathcal{B})$ be some measurable space and let $f \in L^2(T^p)$ and $g \in L^2(T^q)$ be two symmetric functions. Prove the following product rule for multiple stochastic integrals:

$$\forall 1 \leq r \leq p \wedge q: \qquad I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g).$$

Exercise 3: Hermite polynomials (10 points)

For every $n \in \mathbb{N}_0$ define the Hermite polynomial $H_n(\lambda, x) := \lambda^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{\lambda}}\right)$ for $x \in \mathbb{R}$ and $\lambda > 0$.

i) Check that $\exp\left(tx - \frac{t^2\lambda}{2}\right) = \sum_{n=0}^{\infty} t^n H_n(\lambda, x)$.

ii) Let W be an isonormal Gaussian process on the Hilbert space $H := L^2(T, \mathcal{B}, \mu)$. Show that

$$H_m\left(\|h\|_H^2, W(h)\right) = \frac{1}{m!} I_m\left(h^{\otimes m}\right)$$

for any $h \in H$.

iii) The classical case. Let $(W_t)_{t\geq 0}$ be a one-dimensional Brownian motion. Show that the process $(H_n(t, W_t))_{t\geq 0}$ is a martingale.

Exercise 4 : Wiener chaos expansion for Brownian functionals (10 points)

By iteration of Itô's representation formula for Brownian functionals and using the connection

$$I_n(f_n) = n! \int_0^{\infty} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n}$$

between the multiple and the conventional Itô stochastic integral for symmetric square integrable functions in the classical situation $\left\{W(h) = \int_{\mathbb{R}_+} h_s dW_s, h \in L^2(\mathbb{R}_+)\right\}$ show that any random variable $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ (where $\mathcal{F}$ is generated by W) can be expressed as an infinite sum of orthogonal multiple stochastic integrals.