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Markov Processes

Exercise sheet 7 from 05/29/2009

Exercise 1: SPDE with generator of log**-normal diffusions** (10 points)

Consider on $\mathbb{R}^n$ the SDE

$$d\gamma_t^x = bdt + \sigma dB_t,\ \gamma_0^x = x$$

with $b \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n\times d}$ and a d-dimensional Brownian motion $(B_t)_{t\geq 0}$.

i) Show that $(S_t u)(x) := \mathbb{E}[u(\gamma_t^x)]$ defines a semigroup on $L^2(\mathbb{R}^n)$.

ii) Calculate the generator A of this semigroup.

iii) Let $(W_t)_{t\geq 0}$ a Q-Wiener process on $L^2(\mathbb{R}^n)$ where Q is a trace class covariance operator and ζ an $\mathcal{F}_0$-measurable $L^2(\mathbb{R}^n)$-valued random variable. Show that the SPDE

$$dX_t = AX_t dt + dW_t,\ X_0 = \zeta$$

admits a unique weak solution in $L^2(\mathbb{R}^n)$.

iv) Find an expression for the solution that is as explicit as possible.

Exercise 2: SPDE as scaling limit of systems of interacting Brownian particles (10 points)

Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion, $\sigma \in \mathbb{R}^{n\times d}$ a matrix and $\rho : (-1,1]^n \times (-1,1]^n \to \mathbb{R}$ a bounded measurable function.

i) Consider for every $N \in \mathbb{N}$ the following system of stochastic differential equations on $\mathbb{R}^n$:

$$dX_t^i = \sigma dW_t + \frac{1}{(2N)^n}\sum_{j\in T_N} X_t^j \rho(i,j)dt,\ X_0^i = i \in T_N \tag{1}$$

where $T_N := (\frac{1}{N}\mathbb{Z} \cap (-1,1])^n$. Show that this system admits a unique solution.

ii) In the limit $N \uparrow \infty$ equation (1) turns into the system

$$dX_t^x = \sigma dW_t + \left[\int_{T_\infty} X_t^y \rho(x,y)dy\right] dt,\ X_0^x = x \in T_\infty$$

where $T_\infty := (-1,1]^n$. Rewrite this system as an SPDE in an appropriate Hilbert space and study existence and uniqueness of solutions.

Exercise 3: Stochastic wave equation (10 points)

Let $D \subset \mathbb{R}^n$ be open and bounded and Δ the Laplacian operator with zero boundary conditions. Denote by $\mathcal{D}((-\Delta)^{-\frac{1}{2}})$ the domain of $(-\Delta)^{-\frac{1}{2}}$, which is the completion of $L^2(D)$ with respect to the norm $\|x\|_{\mathcal{D}((-\Delta)^{-\frac{1}{2}})} := \|(-\Delta)^{-\frac{1}{2}}x\|_{L^2(D)}$. Define the Hilbert space $H := L^2(D) \oplus \mathcal{D}((-\Delta)^{-\frac{1}{2}})$, provided with the scalar product $\langle (y,z),(y_1,z_1)\rangle_H := \langle y,y_1\rangle_{L^2(D)} + \langle (-\Delta)^{-\frac{1}{2}}z, (-\Delta)^{-\frac{1}{2}}z_1\rangle_{L^2(D)}$. Consider on H the so called stochastic wave equation

$$dX_t = AX_t dt + BdW_t,\ X_0 = \zeta \tag{2}$$

where A is an operator on $\mathcal{D}(A) := \mathcal{D}((-\Delta)^{\frac{1}{2}}) \oplus L^2(D) \subset H$, defined by $A(y,z) := (z, \Delta y)$. Furthermore let $(W_t)_{t\geq 0}$ a Q-Wiener process on $\mathcal{D}((-\Delta)^{-\frac{1}{2}})$, $Bu := (0,u)$ for any $u \in \mathcal{D}((-\Delta)^{-\frac{1}{2}})$ and ζ an $\mathcal{F}_0$-measurable H-valued random variable.

i) Justfify the name of the above SPDE (2).

ii) Show that the semigroup $(S_t)_{t\geq 0}$ generated by A is given by

$$S_t \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos((-\Delta)^{\frac{1}{2}} t) & (-\Delta)^{-\frac{1}{2}} \sin((-\Delta)^{\frac{1}{2}} t) \\ -(-\Delta)^{\frac{1}{2}} \sin((-\Delta)^{\frac{1}{2}} t) & \cos((-\Delta)^{\frac{1}{2}} t) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

iii) Conclude that the stochastic convolution is given by

$$W_A(t) = \left((-\Delta)^{-\frac{1}{2}} \int_0^t \sin((-\Delta)^{\frac{1}{2}} (t-s)) dW_s, \int_0^t \cos((-\Delta)^{\frac{1}{2}} (t-s)) dW_s \right).$$

iv) Under which conditions on the set D does equation (2) admit a unique weak solution in H? Consider both the cases where Q is trace class and where $Q = \mathrm{Id}$.

Exercise 4 : Retardation equation (10 points)

Consider on $\mathbb{R}^n$ the so called stochastic retardation equation

$$(3)\qquad \begin{aligned} dz_t &= \left[\int_{-r}^0 a(d\theta) z_{t+\theta} \right] dt + f_t dt + dW_t \\ z_0 &= h_0 \\ z_\theta &= h_1(\theta) \quad ,\theta \in [-r, 0]. \end{aligned}$$

Herein let $r > 0$ be the maximal retardation, a an $\mathbb{R}^{n\times n}$-valued finite mass on $[-r, 0]$, $f : \mathbb{R}_+ \to \mathbb{R}^n$ locally integrable, $h_0 \in \mathbb{R}^n, h_1 \in L^2([-r, 0], \mathbb{R}^n)$ and $(W_t)_{t\geq 0}$ an n-dimensional Brownian motion.

i) Show that equation (3) is equivalent to the following stochastic differential equation (with "finite noise") on the Hilbert space $H := \mathbb{R}^n \oplus L^2([-r, 0], \mathbb{R}^n)$:

$$(4)\qquad dX_t = AX_t dt + Bf_t dt + BdW_t,\ X_0 = (h_0, h_1)$$

where $A(z, f) := (\int_{-r}^0 a(d\theta) f(\theta), \frac{df}{d\theta})$, $(z, f) \in H$ and $Bz := (z, 0), z \in \mathbb{R}^n$.

ii) Show that equation (4) admits a unique weak solution in H (you can use without a proof that A generates a strongly continuous semigroup on H).

iii) Calculate explicitly the solution in the case $a := a_0\delta_0 + a_1\delta_{-r}$ with $a_0, a_1 \in \mathbb{R}^{n\times n}$.