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Markov Processes

Exercise sheet 6 from 05/22/2009

Exercise 1: Heat equation (10 points)

Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open set. Consider the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) \text{ on } \mathbb{R}_+ \times \mathcal{O} \\ u(0,x) = f(x),\ x \in \mathbb{R}^n \end{cases}$$

i) Let $\mathcal{O} = \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$ and $u(x,t)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ be a solution of the heat equation. Assume furthermore that u is twice continuously differentiable in x, once continuously differentiable in t and satisfies $\|u\|_{L^1(\mathbb{R}^n)} + \|\Delta u\|_{L^1(\mathbb{R}^n)} < \infty$. Show that the Fourier transformation of this solution is given by $\hat{u}(t,x) = e^{-|x|^2 t} \hat{f}(x)$. Use this to show that the solution can be expressed as

$$u(t,x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2t}|x-y|^2} f(y) dy.$$

ii) Let $\mathcal{O}$ be bounded. Define for $t \geq 0$ the operator $e^{t\Delta} := \sum_{k=0}^{\infty} \Delta^k \frac{t^k}{k!} : \mathcal{C}^\infty(\mathcal{O}) \to \mathcal{C}^\infty(\mathcal{O})$. Show that the operator exists and is continuous, and that for $f \in \mathcal{C}^\infty(\mathcal{O})$, $u(t,x) := e^{t\Delta} f(x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ is a solution to the heat equation.

iii) Let $\mathcal{O}$ be bounded, $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of $-\Delta$, $\langle \bullet, \bullet \rangle$ the $L^2(\mathbb{R}^n)$-scalar product and $f \in L^2(\mathbb{R}^n)$. Solve the heat equation by using the ansatz $u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k$. Show that this u coincides with $e^{\bullet \Delta} f$ from the previous part.

Exercise 2 : Characterization of Hilbert spaces (10 points)

Consider the space $l^2(\mathbb{R})$ of square-summable sequences,

$$l^2(\mathbb{R}) := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=0}^{\infty} x_n^2 < \infty\}$$

equipped with the scalar poduct

$$(x,y)_{l^2(\mathbb{R})} := \sum_{n=0}^{\infty} x_n y_n,\ x, y \in l^2(\mathbb{R}).$$

i) Show that $\left(l^2(\mathbb{R}), (x,y)_{l^2(\mathbb{R})}\right)$ is a Hilbert space by finding a suitable measure μ such that $l^2(\mathbb{R}) = L^2(\mathbb{R}, \mu)$

ii) Prove the
Theorem: Every separable Hilbert space H is isometrically isomorph to $l^2(\mathbb{R})$.

iii) Show that for every separable Hilbert space H there exists a Hilbert space $H_1 \supseteq H$ such that the inclusion operator $j : H \to H_1$ is a Hilbert-Schmidt operator.
Hint: Fix a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^2(\mathbb{R}) \cap \mathbb{R}_+^{\mathbb{N}}$ and consider the space

$$H_1 := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=0}^{\infty} \lambda_n x_n^2 < \infty\}.$$

iv) Let $\{e_n\}_{n\in\mathbb{N}}$ be the Fourier basis of $L^2([0,1])$, and $(\lambda_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. Identify the space H_1 of the previous part with the Soboleff space $H^{-1}([0,1])$.

Exercise 3: Cylindrical Wiener process (10 points)

Let U, H be separable Hilbert spaces.

i) Let $\{W_t\}_{t\geq 0}$ be a Q-Wiener process on U and $B \in L(U; H)$. Show that $\left\{\tilde{W}_t\right\}_{t\geq 0} := \{BW_t\}_{t\geq 0}$ is a BQB^*-Wiener process in H.

ii) Extend this result to cylindrical Wiener processes and $B \in L_2(U; H)$. Why is $\left\{\tilde{W}_t\right\}_{t\geq 0}$ well-defined?

iii) If Q is bounded and B is a Hilbert-Schmidt-operator, show that BQB^* is trace class.

Exercise 4: Estimate for a strong solution of an SPDE (10 points)

Let $(H, \langle \bullet, \bullet \rangle_H)$ be a separable Hilbert space. Let the stochastic process $\{X_t\}_{t\geq 0}$ be a solution of the integral equation

$$X_t := x + \int_0^t A(X_t)dt + BW_t$$

where $A, B \in L(H)$ are linear operators, $\{W_t\}_{t\geq 0}$ is a Q-Wiener process (Q is as usual trace class, symmetric and positive definite) and $x \in H$. Assume that $\mathbb{E}|X(t)|_H^2 < \infty$, $\mathbb{E}|AX(t)|_H^2 < \infty$ for all $t \geq 0$.

i) Show the identity

$$\frac{1}{2}\mathbb{E}|X(t)|_H^2 = \frac{1}{2}|x|_H^2 + \mathbb{E}\int_0^t \langle X_s, AX_s\rangle_H ds + \frac{1}{2}\text{Tr}(BQB^*)t, \ t \geq 0.$$

ii) Use $i)$ to obtain for all $T > 0$ the estimate

$$\frac{1}{2}\mathbb{E}|X(t)|_H^2 \leq \frac{1}{2}\left(|x|_H^2 + \text{Tr}(BQB^*)T\right) e^{(1+\|A\|^2)t}, \ t \in [0, T].$$