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Markov Processes

Exercise sheet 5 from 05/15/2009

In the following let $(H, \langle \cdot, \cdot \rangle)$ be an infinite dimensional separable Hilbert space and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis for H.

Exercise 1: Projective expectations (10 points)

Consider an H-valued random variable X that is concentrated on the set $\{ke_k | k \in \mathbb{N}\}$ with $\mathbb{P}[X = ke_k] = C\frac{1}{k^2}$ and $C := \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{-1}$. Prove that X is not integrable, although for every $h \in H$ we have $\mathbb{E}[\langle X, h\rangle^2] < \infty$.

Exercise 2 : Hilbert-Schmidt operators (10 points)

Definition: A linear bounded operator $T : H \to H$ is called *Hilbert-Schmidt operator* iff $\sum_{k=1}^{\infty} \|Te_k\|_H^2 < \infty$. Denote the set of all Hilbert-Schmidt operators on H by $L_2(H)$. Then $L_2(H)$ is again a Hilbert space, equipped with the scalar product $\langle S, T\rangle_{L_2(H)} := \sum_{k=1}^{\infty} \langle Se_k, Te_k\rangle_H$. Now let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of real numbers and define a linear symmetric operator $T : H \to H$ by the formula

$$Th := \sum_{k=1}^{\infty} \lambda_k \langle h, e_k\rangle e_k.$$

Show that

i) T is bounded iff $\sup_{k\in\mathbb{N}} |\lambda_k| < \infty$

ii) T compact iff $\lim_{k\to\infty} \lambda_k = 0$

iii) T is trace class iff $\sum_{k\in\mathbb{N}} |\lambda_k| < \infty$

iv) $T \in L_2(H)$ iff $\sum_{k\in\mathbb{N}} |\lambda_k|^2 < \infty$.

Exercise 3: Construction of Q-Wiener process (10 points)

Let $\left(\beta_t^k\right)_{t\geq 0}$, $k \in \mathbb{N}$ be a sequence of independent one-dimensional Brownian motions and Q a nonnegative symmetric trace class operator on H such that for every $k \in \mathbb{N}$ the element e_k is an eigenvector of Q with eigenvalue λ_k. Show that

$$X_t := \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_t^k e_k,\ t \geq 0$$

exists and defines a Q-Wiener process on H.

Exercise 4: Projections of Q-Wiener process (10 points)

Let Q be a nonnegative symmetric trace class operator on H and $X : \Omega \times \mathbb{R}_+ \to H$ a Q-Wiener process. For $h \in H$ define the real-valued process $X_t^h := \langle X_t, h\rangle$. Show that X is a centered Gaussian process and compute its covariance function. Moreover determine the quadratic covariation for X^h and $X^{h'}$.