Institut für angewandte Mathematik
Universität Bonn
Prof. Dr. K.-Th. Sturm
Frank Miebach
Bernhard Hader
http://www-wt.iam.uni-bonn.de/~sturm/de/ss09.html

Markov Processes

Exercise sheet 4 from 05/08/2009

Exercise 1: Nuclear Operators (10 points)

Let X, Y be Banach spaces. A continuous linear Operator $T \in L(X,Y)$ is called *nuclear* iff one can find sequences $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ with $\sum_{n=1}^\infty \|x_n^*\| \, \|y_n\| < \infty$ such that for all $x \in X$

$$Tx = \sum_{n=1}^\infty x_n^*(x) y_n < \infty.$$

Denote the set of all nuclear operators from X to Y with $L_1(X,Y)$, and define for $T \in L_1(X,Y)$ the *nuclear norm* by

$$\|T\|_{L_1} := \inf \left\{ \sum_{n=1}^\infty \|x_n^*\| \, \|y_n\| : \; Tx = \sum_{n=1}^\infty x_n^*(x) y_n \right\}.$$

i) Show that $\left(L_1(X,Y), \|\bullet\|_{L_1}\right)$ is a Banach space.

ii) Let W, Z be additional Banach spaces, $R \in L(W,X)$, $S \in L(X,Y)$ and $T \in L(Y,Z)$. Show that $\|TSR\|_{L_1} \leq \|T\| \, \|S\|_{L_1} \|R\|$, hence $TSR \in L_1(W,Z)$. (where $\|\bullet\|$ denotes the usual operator norm.)

iii) Let $(H, \langle \bullet, \bullet \rangle)$ be a separable Hilbert space, $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of H and $T \in L_1(H) := L_1(H,H)$. Show that the series

$$\mathrm{Tr}(T) := \sum_{n=1}^\infty \langle Te_n, e_n \rangle$$

converges absolutely, that $Tr(T)$ is independent of the choice of the orthonormal basis, and that

$$|\mathrm{Tr}(T)| \leq \|T\|_{L_1}.$$

iv) Show that a selfadjoint, positiv semidefinit Operator $T \in L(H)$ is nuclear iff $\mathrm{Tr}(T) < \infty$. Show that in this case $|\mathrm{Tr}(T)| = \|T\|_{L_1}$.

Exercise 2 : Quadratic Variation in Hilbert spaces (10 points)

Let $(H, \langle \bullet, \bullet \rangle)$ be a infinite dimensional, separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, and $\{X_t\}_{t\in[0,T]}$, $T > 0$ be an H-valued, continuous, square integrable martingale. Define the operator-valued random variable $\{V_t\}_{t\in[0,T]}$, $T > 0$ by

$$V_t(h) := \sum_{k,l=1}^\infty \left\langle \left\langle X^k, X^l \right\rangle \right\rangle_t \langle h, e_k \rangle \, e_l, \; h \in H, \; t \in [0,T],$$

where $X_t^k := \langle X_t, e_k \rangle$ and $\left\langle \left\langle X^k, X^l \right\rangle \right\rangle_t$ denotes the quadratic Variation of X_k and X_l.

i) Show that the sum in the definition of V converges a.s., that $V \in L_1(H)$ and that for all $u, w \in H$, $V^{u,w} := \langle Vu, w \rangle$ defines a continuous process with finite variation.

ii) Show that for all $u, w \in H$, the process $\langle X, u \rangle \langle X, w \rangle - V^{u,w}$ is a continuous martingale.

Remark: The result

$$\langle X, u \rangle \langle X, w \rangle - V^{u,w} \text{ is a continuous martingale for arbitrary } u, w \in H$$

can be equivalently written as

$$X \otimes X - V \text{ is a continuous martingale.}$$

So V fulfills the role of a quadratic variation of the H-valued Martingale X, and is denoted by $\langle\langle X \rangle\rangle$.

Exercise 3: A Gaussian process(10 points)

Let $(E, \mathcal{E}, \mu)$ be a measure space and $I := L^2(E, \mathcal{E}, \mu)$. Define for $u, v \in I$

$$\Gamma(u, v) := \int_E u(t)v(t)d\mu(t).$$

i) Show that Γ is positive semidefinite.

ii) Let $\{B_t\}_{t\geq 0}$ be a one-dimensional Brownian Motion. For $u \in L^2(\mathbb{R}_+)$ set $Y_t := \int_0^\infty u(t)dB_t$. Show that $\{Y_u\}_{u\in L^2(\mathbb{R}_+)}$ is a centered Gaussian process with covariance $\mathrm{Cov}(Y_u, Y_v) := \int_0^\infty u(t)v(t)dt$.

Exercise 4: Negative inverse Laplacian (10 points)

The negative Laplace operator $-\Delta$ of a function $u \in H_0^1(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^n$ is an open, bounded subset, can be defined by

$$(-\Delta u, v) := \int_{\mathcal{O}} \nabla u(x)\nabla v(x)dx \text{ for all } v \in H_0^1(\mathcal{O}). \tag{1}$$

Here $H_0^1(\mathcal{O}) := \left\{u \in H^1(\mathcal{O}) | \exists \{f_k\}_{k\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathcal{O}) : \lim_{k\to\infty} |f - f_k|_{H^1(\mathcal{O})} = 0\right\}$
is equipped with the norm $|u|_{H_0^1(\mathcal{O})} := |\nabla u|_{L^2(\mathcal{O})}$, $u \in H_0^1(\mathcal{O})$, where ∇ is the weak gradient.

i) Show that $-\Delta u$ defines for every $u \in H_0^1(\mathcal{O})$ an element of $H^{-1}(\mathcal{O}) := H_0^1(\mathcal{O})^*$

ii) Show that the Laplacian constructed by (1) is invertible, i.e. for arbitrary $f \in H^{-1}(\mathcal{O})$, the problem $-\Delta u = f$, has a unique solution $u \in H_0^1(\mathcal{O})$.
Hint: For surjectivity, you can show that the range of Δ is closed and construct a functional $u_0 \in H^{-1}(\mathcal{O})^*$ that vanishes on Range(Δ). Then use the reflexivity of $H_0^1(\mathcal{O})$.

iii) With Soboleff and Rellich embedding theorems it can be shown that $(-\Delta)^{-1}$, restricted to the space $L^2(\mathcal{O})$, is a linear, compact operator. Show that $(-\Delta)^{-1}|_{L^2(\mathcal{O})}$ is selfadjoint and positive definite on $L^2(\mathcal{O})$.

iv) Now consider $\mathcal{O} := (0, R)^n$, $R > 0$. Calculate for this special $\mathcal{O}$ the eigenvalues and the trace of $(-\Delta)^{-1}$ and determine for which R and n the negative inverse Laplacian is trace class. You can use that an orthonormal basis of $L^2((0, R)^n)$ is given by the functions

$$e_{(k_1,\dots,k_n)}(x) := \left(\frac{2}{R}\right)^{\frac{d}{2}} \prod_{i=1}^n \sin\left(\frac{k_i\pi}{R}x_i\right), \ (k_1, \dots, k_n) \in \mathbb{N}^n.$$