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Markov Processes

Exercise sheet 3 from 05/01/2009

Exercise 1: Multiple Stratonovich Integrals (10 points)
Let $m \in \mathbb{N}$. We denote the set of all multi-indices of length $l \geq 0$ by

$$\mathcal{M}_l := \{(j_1 \dots, j_l) : \ j_i \in \{0, 1, \dots, m\}, \ i \in \{0, 1, \dots, l\}\}$$

We write $\alpha-$ for the multi-index in $\mathcal{M}_{l-1}$ obtained by deleting the last component of $\alpha \in \mathcal{M}_l$.
Define the Multiple Stratonovich Integrals recursively by

$$J_{\alpha,t} := \begin{cases} 1, \ l = 0 \\ \int_0^t J_{\alpha-,s} * dW_s^{j_l}, \ l \geq 1, \ j_l \geq 1, \end{cases}$$

where $\left\{W_t^i\right\}_{t \geq 0}$, $i = 1, \dots, m$ are standard Brownian Motions, and $W_t^0 := t, \ t \geq 0$.

i) Let $\alpha \in \mathcal{M}_l$. Show that

$$W_t^j J_{\alpha,t} = \sum_{i=0}^{l} J_{(j_1,\dots,j_i,j,j_{i+1},\dots,j_l),t}.$$

ii) Let $\{W_t\}_{t \geq 0}$ be a standard Brownian Motion. Show that

$$\int_0^t \int_0^{t_l} \dots \int_0^{t_2} *dW_{t_1} * dW_{t_2} \dots * dW_{t_l} = \frac{1}{l!} [W_t]^l = \frac{1}{l!} \left[\int_0^t *dW_s\right]^l.$$

And now for something completely different...
Suppose in the following exercises that $\sigma \in \mathcal{C}^2(\mathbb{R})$ with bounded first and second derivatives, and that b is Lipschitz continuous. Let $\{W_t\}_{t \geq 0}$ be a one-dimensional Brownian Motion and $\left(\left\{V_t^{(n)}\right\}_{t \geq 0}\right)_{n=1}^{\infty}$ a sequence of processes such that for all $n \in \mathbb{N}$ the following properties hold:

- $\mathbb{P}$-a.e. path $V_\bullet^{(n)}(\omega)$ is continuous and has finite total variation on compact intervals.
- For every $t \geq 0$ we have

$$\lim_{n \to \infty} \sup_{0 \leq s \leq t} \left|V_s^{(n)} - W_s\right| = 0, \ \text{a.s.}$$

Consider the Stratonovich-equation

$$dX_t = b(X_t)dt + \sigma(X_t) * dW_t, \ X_0 = x; \ t \geq 0 \tag{1}$$

and the stochastic Integral equation

$$X_t^{(n)} = x + \int_0^t b(X_s^{(n)})ds + \int_0^t \sigma(X_s^{(n)})dV_s^{(n)} \tag{2}$$

Our goal is to show that the solutions of (2) exist uniquely and converge almost surely, uniformly on bounded intervals to the unique solution of (1).

Exercise 2 : Solution for a Stratonovich-equation II (10 points)

i) Let $u(x,z)$ be a solution of the equation

$$\frac{\partial}{\partial x}u(x,z) = \sigma\left(u(x,z)\right);\ u(0,y) = y. \tag{3}$$

Show that the function

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto \exp\left(-\int_0^x \sigma'\left(u(z,y)\right)dz\right) b\left(u(x,y)\right)$$

satisfies the Lipschitz and linear growth condition

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &\le L_k\,|y_1 - y_2|\,;\ -k \le x, y_1, y_2 \le k, \\ |f(x,y)| &\le K_1 + K_k|y|;\ |x| \le k, y \in \mathbb{R}. \end{aligned}$$

ii) Let $Y_t(\omega)$ be a solution of the ODE

$$\frac{d}{dt}Y_t(\omega) = f\left(W_t(\omega), Y_t(\omega)\right);\ Y_0 = x, \tag{4}$$

for any $\omega \in \Omega$ (exists uniquely by part i)). Show that the process

$$X_t := u\left(W_t, Y_t\right);\ t \ge 0$$

solves the Stratonovich-equation (1).

Exercise 3: Solution of the integral equation (2)(10 points)

Show that for every $n \in \mathbb{N}$, the equation (2) has a unique solution.
(*Hint:* Use Picard-Lindelöff iteration scheme as in the proof of the existence and uniqueness of a solution to an SDE. Note that convergence of the Picard-sequence to the solution is much easier than in the case of noise with infinite variation.)

So what remains to be shown is the convergence of the solutions of (2) to the solution of (1), which will be done in the next exercise.

Exercise 4: Proof of the convergence (10 points)

Show that the sequence of unique solutions for (2) converges almost surely, uniformly on bounded intervals to the unique solution of (1), that is

$$\lim_{n\to\infty}\sup_{0\le s\le t}\left|X_s^{(n)} - X_s\right| = 0,\ \text{a.s.} \tag{5}$$

for all $t \ge 0$.

Steps of the proof:

i) Proof that, in analogy to exercise 2, the solution $X^{(n)}$ to the integral equation (2) can be written as

$$X_t^{(n)} := u\left(V_t^{(n)}, Y_t^{(n)}\right),\ t \ge 0$$

for u defined by (3) and $Y^{(n)}$ as in (4) with W replaced by $V^{(n)}$, and that it is sufficient to show

$$\lim_{n\to\infty}\sup_{0\le s\le t}\left|Y_s^{(n)} - Y_s\right| = 0,\ \text{a.s.} \tag{6}$$

for all $t \ge 0$ to get (5).

ii) Define for $n \in \mathbb{N}$ the stopping times

$$\begin{aligned} \tau_k &: = t \wedge \inf\{s \in [o,t] : |Y_s| \ge k-1 \text{ or } |W_s| \ge k-1\} \\ \tau_k^{(n)} &: = t \wedge \inf\{s \in [0,t] : |Y_s^{(n)}| \ge k\}. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Show that for large $n = n(\epsilon)$ and $s \in \left[0, \tau_k \wedge \tau_k^{(n)}\right]$ we have

$$\left|\frac{d}{ds}\left(Y_s^{(n)} - Y_s^{(n)}\right)\right| \le L_k\left|Y_s^{(n)} - Y_s^{(n)}\right| + \varepsilon^2.$$

Then apply Gronwall's lemma to get (6).