INSTITUT FÜR ANGEWANDTE MATHEMATIK
UNIVERSITÄT BONN
Prof. Dr. K.-Th. Sturm
Frank Miebach
Bernhard Hader
http://www-wt.iam.uni-bonn.de/∼sturm/de/ss09.html

Markov Processes

Exercise sheet 2 from 04/24/2009

Exercise 1: "Differential properties" of the Stratonovich-integral (10 points)

Let $\{X_t\}_{t\geq 0} = \left\{\left(X_t^{ij}\right)\right\}_{t\geq 0}$ be an $M_{n,l}(\mathbb{R})$-valued continuous semimartingale, and $\{Y_t\}_{t\geq 0} = \left\{\left(Y_t^{ij}\right)\right\}_{t\geq 0}$ be an $M_{l,m}(\mathbb{R})$-valued continuous semimartingale. The Ito- and the Stratonovich-integral in this case are given by

$$\left(\int_0^t X_s dY_s\right)^{ij} := \sum_{k=1}^{l} \left(\int_0^t X_s^{ik} dY_s^{kj}\right), t \geq 0$$

$$\left(\int_0^t X_s * dY_s\right)^{ij} := \sum_{k=1}^{l} \left(\int_0^t X_s^{ik} dY_s^{kj} + \frac{1}{2}\langle X^{ik}, Y^{kj}\rangle_t\right), t \geq 0.$$

Show that the Stratonovich-integral has the following properties:

i)

$$d(XY)_t = X_t * dY_t + \left(Y_t^T * dX_t^T\right)^T, t \geq 0 \text{ (product rule)},$$

where XY denotes the usual matrix product and X^T the transposed of X.

ii)

$$df(X_t) = \sum_{i=1}^{n}\sum_{j=1}^{l} \partial_{(i,j)} f(X_t) * dX_t^{ij}, t \geq 0 \text{ (chain rule)},$$

where $f \in \mathcal{C}^3(M_{n,l}(\mathbb{R}))$, and $\partial_{(i,j)}$ denotes the partial derivative of the (i,j)th coordinate.

iii) Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be n-dimensional continuous semimartingales, $(\bullet, \bullet)$ the scalar product in $\mathbb{R}^n$ and $f \in \mathcal{C}^3(\mathbb{R}^n)$. By using the parts i) and ii), find nice expressions in Stratonovich calculus for $d(X,Y)$ and $df(X)$.

Exercise 2: Stratonovich integral as limit in probability (10 points)

Let $\{V_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ be continuous semimartingales and $\Delta := \{t_0, \ldots, t_N\}$ be a partition of $[0,t]$ with $0 = t_0 < t_1, \cdots < t_N \leq t$.

i) Show that

$$\int_0^t Z_s * dV_s = \text{l.i.p.}_{\|\Delta\|\downarrow 0} \sum_{i=0}^{N} \frac{1}{2}\left(Z_{t_{i+1}} + Z_{t_i}\right)\left(V_{t_{i+1}} - V_{t_i}\right),$$

where l.i.p. denotes the limit in probability.

ii) Show that, if in addition $d\langle Z, V\rangle$ is absolutely continuous w.r.t. the Lebesgue measure, we get

$$\int_0^t Z_s * dV_s = \text{l.i.p.}_{\|\Delta\|\downarrow 0} \sum_{i=0}^{N} Z_{\frac{t_{i+1}+t_i}{2}}\left(V_{t_{i+1}} - V_{t_i}\right).$$

Exercise 3: Solution for a Stratonovich-equation (10 points)

i) Let $u = (u_1, \ldots, u_n) \in \mathcal{C}^2([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ be a solution of the equations

$$\frac{\partial}{\partial t} u_i(t,x) = b_i(t, u(t,x)), \ \frac{\partial}{\partial x_j} u_i(t,x) = \sigma_{ij}(t, u(t,x)); \ 1 \le i,j \le n$$

on $[0,\infty) \times \mathbb{R}^n$, where $b = (b_1, \ldots, b_n) \in \mathcal{C}([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ and $\sigma = (\sigma_{ij})_{1\le i,j \le n} \in \mathcal{C}^2([0,\infty) \times \mathbb{R}^n, M_n(\mathbb{R}))$. Let $\{W_t\}_{t\ge 0}$ be a d-dimensional Brownian Motion. Show that the process

$$X_t := u(t, W_t); \ t \ge 0$$

solves the Stratonovich-equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) * dW(t); \ t \ge 0.$$

ii) Determine the unique, strong solution of the one-dimensional Ito-equation

$$dX_t = \left[\frac{2}{1+t} X_t - a(1+t)^2\right] dt + a(1+t)^2 dW_t; \ t \ge 0$$

by using the results of part i) of this exercise.

Exercise 4: Ornstein-Uhlenbeck Feller semigroup (10 points)

Let $\{P_t\}_{t\ge0}$ be a set of operators on $\mathcal{C}_0(\mathbb{R}) := \left\{f \in \mathcal{C}(\mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0\right\}$, defined by

$$P_t f(x) := \mathbb{E} f\left(e^{-\frac{t}{2}} x + \sqrt{1-e^{-t}} Z\right), \ f \in \mathcal{C}_0(\mathbb{R}), \ x \in \mathbb{R}, \ t \ge 0,$$

where Z is an $\mathcal{N}(0,1)$-distributed random variable. Show that $\{P_t\}_{t\ge0}$ is a Feller semigroup.