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Markov Processes

Exercise sheet 11 from 07/03/2009

Exercise 1: Hörmander-condition (10 points)
Let $A_0, \ldots, A_d$ be smooth vector fields on $\mathbb{R}^m$, and $\{W_t\}_{t\geq 0}$ a d-dimensional Brownian Motion. Consider the Stratonovich-SDE

$$dX_t = A_0(X_t)dt + \sum_{j=1}^{d} A_j(X_t) \circ dW_t^j.$$

We know from the lecture that the associated generator is given by $\mathcal{L} = \frac{1}{2}\sum_{j=1}^{d} A_j^2 + A_0$.

i) Prove that the adjoint operator $\mathcal{L}^*$ has the form

$$\mathcal{L}^* = \frac{1}{2}\sum_{j=1}^{d} A_j^2 - \widetilde{A}_0 + c$$

with $\widetilde{A}_0 := A_0 - \sum_{j\leq d, i\leq m} \frac{\partial A_j^i}{\partial x_i} A_j$ and

$$c := \frac{1}{2}\sum_{i,k=1}^{m} \frac{\partial^2}{\partial x_i \partial x_k}\left(\sum_{j=1}^{d} A_j^i A_j^k\right) - \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(A_0^i + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{m} \frac{\partial A_j^i}{\partial x_k} A_j^k\right).$$

ii) Calculate $\mathcal{L}$ and $\mathcal{L}^*$ for the special case $d = 2$, $m = 3$, $A_0 = 0$, $A_1 = (1, 0, -y/2)$, $A_2 = (0, 1, x/2)$.

iii) Does the Hörmander condition hold?

Exercise 2 : Product rule for the divergence (10 points)
Let $F \in \mathbb{D}^{1,2}$ and u in the domain of δ such that $Fu \in L^2(\Omega, H)$. Prove that Fu belongs to the domain of δ and has divergence

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H,$$

provided the right-hand side is square integrable.

In the following you may use the following further properties of the Malliavin derivative without a proof, if necessary:

i) *Covariance relation*: Let $u, v \in \mathbb{D}^{1,2}(H) \subset \mathrm{Dom}(\delta)$. Then

$$\mathbb{E}[\delta(u)\delta(v)] = \mathbb{E}\left[\langle u, v\rangle_H\right] + \mathbb{E}[\mathrm{Tr}(Du \circ Dv)] \overset{H=L^2(T)}{=} \mathbb{E}\left[\langle u, v\rangle_H\right] + \mathbb{E}\left[\int_T \int_T D_s u_t D_t v_s \mu(ds)\mu(dt)\right].$$

ii) *Commutator relation*: Suppose that $u \in \mathbb{L}^{1,2}$. Assume that for almost all t the process $\{D_t u_s, s \in T\}$ is Skorohod integrable, and there is a version of the process $\{\int_T D_t u_s dW_s, t \in T\}$ which is in $L^2(T \times \Omega)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and we have

$$D_t(\delta(u)) = u_t + \int_T D_t u_s dW_s.$$

Exercise 3: Absolute continuity (10 points)

Let F be a random variable in the space $\mathbb{D}^{1,2}$. Suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator δ. Show that the law of F has a continuous and bounded density given by

$$p(x) = \mathbb{E}\left[\mathbb{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

Hint: Fix $a < b$ and define the function $\psi(z) := \mathbb{1}_{[a,b]}(z)$ to compute $\mathbb{P}(a \leq F \leq b) = \mathbb{E}[\psi(F)]$ by using the integration by parts formula and exercise 3 from sheet 9.

Exercise 4: Sufficient condition (10 points)

Let $H := L^2(T, \mu)$ and let F be a random variable in $\mathbb{D}^{2,4}$ such that $\mathbb{E}\left[\|DF\|_H^{-8}\right] < \infty$. Show that this is sufficient for $\frac{DF}{\|DF\|_H^2}$ to be in the domain of the divergence operator and that we have explicitely

$$\delta\left(\frac{DF}{\|DF\|_H^2}\right) = -\frac{LF}{\|DF\|_H^2} + 2\frac{\left\langle DF \otimes DF, D^2F\right\rangle_{L^2(T^2)}}{\|DF\|_H^4}.$$

where $L := \delta D$ is the Ornstein-Uhlenbeck operator.
Hint: Show first that $\frac{DF}{\|DF\|_H^2+\epsilon}$ belongs to $\mathrm{Dom}(\delta)$ *for any $\epsilon > 0$ using exercise 2 and then let ϵ tend to zero.*
Remark: It is not necessary to assume that H is an L^2-space. The result holds true for every separable Hilbert space where we replace $L^2(T^2)$ by $H \otimes H$ (and the proof actually remains the same).