Solution of Aufgabe 4, Blatt 6

i) First note that for any odd function f the derivative f' is even, i.e. $f(x) = f'(-x)$ holds. This follows because $-f'(-x) = \frac{d}{dx} f(-x) = \frac{d}{dx}(-f(x)) = -f'(x)$.

Moreover, any odd function f fulfills $f(0) = 0$. Consequently, for g as in Aufgabe 4 of Blatt 6 the functions g' and $g^{(3)}$ are even and g'' and $g^{(4)}$ are odd. In particular, $g(0) = 0$, $g''(0) = 0$ and $g^{(4)}(0) = 0$.

From Taylor's formula with "Integral-Restglied" we obtain

$$g(x) = xg'(0) + \frac{1}{3!}x^3 g^{(3)}(0) + \frac{1}{4!}\int_0^x (x-t)^4 g^{(5)}(t)\ dt \tag{1}$$

By applying Taylor's formula with "Integral-Restglied" for g' we get

$$g'(x) = g'(0) + \frac{1}{2!}x^2 g^{(3)}(0) + \frac{1}{3!}\int_0^x (x-t)^3 g^{(5)}(t)\ dt \tag{2}$$

By multiplying equation (2) with $-\frac{x}{3}$ and adding the result to equation (1) one obtains

$$\begin{aligned} g(x) - \frac{x}{3}g'(x) &= xg'(0) - \frac{x}{3}g'(0) + \frac{1}{4!}\int_0^x (x-t)^4 g^{(5)}(t)\ dt - \frac{x}{3}\frac{1}{3!}\int_0^x (x-t)^3 g^{(5)}(t)\ dt \\ &= \frac{2x}{3}g'(0) + \frac{1}{4!}\int_0^x g^{(5)}(t)\ (x-t)^3\big((x-t) - \frac{4x}{3}\big)\ dt \end{aligned}$$

So the assertion follows from

$$\begin{aligned} &\frac{1}{4!}\int_0^x g^{(5)}(t)(x-t)^3\big((x-t) - \frac{4x}{3}\big)\ dt = -\frac{1}{4!}\int_0^x g^{(5)}(t)(x-t)^3\big(\frac{x}{3} + t\big)\ dt \\ &\overset{(*)}{=} -\frac{1}{4!}g^{(5)}(\xi)\ \int_0^x (x-t)^3\big(\frac{x}{3} + t\big)\ dt \overset{(**)}{=} -\frac{1}{4!}g^{(5)}(\xi) \cdot \frac{2}{15}x^5 = -\frac{1}{180}x^5 g^{(5)}(\xi) \end{aligned}$$

for some $\xi \in [0, x]$. Here step $(*)$ follows from the "Mittelwertsatz der Integral-Rechnung" (which is applicable because $(x-t)^3\big(\frac{x}{3} + t\big) \geq 0$ for $t \in [0, x]$). Step $(**)$ follows because

$$\begin{aligned} \int_0^x (x-t)^3\big(\frac{x}{3} + t\big)\ dt &= \int_0^x (x-t)^3 \frac{x}{3}\ dt + \int_0^x (x-t)^3 t\ dt \\ &= \frac{x}{3}\big[-\frac{1}{4}(x-t)^4\big]_{t=0}^x + \big[-\frac{1}{4}(x-t)^4 t\big]_{t=0}^x + \int_0^x \frac{1}{4}(x-t)^4\ dt \\ &= \frac{1}{12}x^5 + 0 + \frac{1}{4}\big[-\frac{1}{5}(x-t)^5\big]_{t=0}^x = (\frac{1}{12} + \frac{1}{20})x^5 = \frac{2}{15}x^5 \end{aligned}$$

ii) Define h and g (on the obvious domains) by

$$h(x) := f(x + \tfrac{a+b}{2})$$
$$g(x) := h(x) - h(-x)$$

Clearly, g is odd, so with $x := \frac{b-a}{2}$ we obtain from part i)

$$f(b) - f(a) = g(\tfrac{b-a}{2}) = \frac{\frac{b-a}{2}}{3}(g'(\tfrac{b-a}{2}) + 2g'(0)) - \frac{(\frac{b-a}{2})^5}{180} g^{(5)}(\xi')$$
$$\overset{(*)}{=} \tfrac{b-a}{6}(f'(a) + f'(b) + 4f'\big(\tfrac{a+b}{2}\big)) - \tfrac{(b-a)^5}{2880}(\tfrac{1}{2}f^{(5)}(\xi' + \tfrac{a+b}{2}) + \tfrac{1}{2}f^{(5)}(-\xi' + \tfrac{a+b}{2}))$$

with some $\xi' \in]0, \frac{b-a}{2}[$. Here step $(*)$ follows because $g'(x) = f'(x + \frac{a+b}{2}) + f'(-x + \frac{a+b}{2})$ and $g^{(5)}(x) = f^{(5)}(x + \frac{a+b}{2}) + f^{(5)}(-x + \frac{a+b}{2})$. From the fact that $f^{(5)}$ is continuous it then follows that there is a $\xi \in]-\xi' + \frac{a+b}{2}, +\xi' + \frac{a+b}{2}[=]a, b[$ (here we have assumed without loss of generality that $\xi' \geq 0$) such that

$$\tfrac{1}{2}f^{(5)}(\xi' + \tfrac{a+b}{2}) + \tfrac{1}{2}f^{(5)}(-\xi' + \tfrac{a+b}{2}) = f^{(5)}(\xi)$$

holds.