From Fokker-Planck to SDEs and back via the martingale problem

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First lecture: Tue Oct 27 due to the Inauguration of the Hausdorff School*

Synopsis: The lecture investigates partial differential equations of parabolic type

$$\partial_t \mu_t = L_t \mu_t := \frac{1}{2} \sum_{i,j=1}^{n} \partial_{ij} \left((\sigma \sigma^T)_{ij} \mu_t \right) - \sum_{i=1}^{n} \partial_i \left(b_i \mu_t \right), \tag{PDE}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ is some vector-field, $\sigma : \mathbb{R}^n \to \mathbb{R}^k$ is a semi-positive definite dispersion matrix with $k \in \mathbb{N}$ and not necessarily of full rank. This type of partial differential equations is very closely related to stochastic differential equations of the form

$$dX_t = b(X_t) dt + \sigma dW_t, \tag{SDE}$$

where W_t is a k-dimensional Brownian motion.

The lecture starts with the smooth case, recalling the solution concepts for (PDE) and (SDE) and the relation between them. That is the law of X_t is given by μ_t. Typical assumptions are of the form:

(PDE)	(SDE)
⊖ strict ellipticity of $\sigma\sigma^T$	⊕ Lipschitz continuity of σ (no ellipticity)
⊕ Hölder estimate on $\sigma\sigma^T$ and b	⊖ (one-sided) Lipschitz bound on b

This raises the question whether solutions can be carried over from one side to the other under weaker assumptions. The concept used for this can be found in martingale solutions introduced by Stroock and Varadhan [1], which will be central for the lecture.

On the one hand, possible applications comprise of establishing solutions to (PDE) without ellipticity assumption on $\sigma\sigma^T$. On the other hand, well-posedness of (PDE) with irregular vectorfield b can be used to proof uniqueness (in law) for weak solutions of (SDE) or to show pathwise uniqueness if b is a bit more regular.

Prerequisits: The lecture addresses students interested in the connection of analysis and stochastic. It tries to be mainly self-consistent only requiring basic knowledge in the areas. Nevertheless, a solid background in one of the areas is beneficial to understand the concepts in the other one.

[1] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffussion Processes*. Springer, 1979.

*http://www.hcm.uni-bonn.de/events/eventpages/2015/hausdorffschool2015/