Functional Analysis

WS 2015/2016
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Problem Sheet 6.

Due 11.12.2015.

Problem 1. (Positive part, absolute value, maximum of two functions) (6+2+2 Points)

Suppose $U \subset \mathbb{R}^n$ is open.

a) Prove that if $u \in W^{1,1}(U)$ then $u^+ \in W^{1,1}(U)$ where $u^+(x) = \max\{u(x), 0\}$ and show that for all $i = 1, \ldots, n$ and a.e. $x \in U$

$$\partial_i u^+(x) = \begin{cases} \partial_i u(x) & \text{if } u(x) > 0, \\ 0 & \text{else.} \end{cases}$$

Prove that also $|u| \in W^{1,1}(U)$.
Hint: Show that there exist functions $h_k \in C^1(\mathbb{R})$ with $|h_k(t) - t^+| < \frac{1}{k}$, $|h_k'| \leq 1$ and $h_k'(t) \to 0$ if $t \leq 0$ while $h_k'(t) \to 1$ if $t > 0$.

b) Suppose $u, v \in W^{1,1}(U)$. Prove that also $\max\{u, v\}, \min\{u, v\} \in W^{1,1}(U)$ where $\max\{u, v\}(x) = \max\{u(x), v(x)\}$ and $\min\{u, v\}(x) = \min\{u(x), v(x)\}$. Compute the weak derivatives.

c) Suppose $E \subset U$ is measurable, $u \in W^{1,1}(U)$ and $u = 0$ a.e. on E. Prove that $Du = 0$ a.e. on E.

Problem 2. (Poincaré inequality) (5+5 Points)

a) Let $U \subset \mathbb{R}^n$ be open and bounded. Show that there exists a constant $C > 0$ such that $\|u\|_{L^p} \leq C \|Du\|_{L^p}$ for all $p \in [1, \infty)$ and all $u \in W_0^{1,p}(U)$.

   Hint: Assume first that $u \in C_c^\infty(U)$.

b) Show that such a constant also exists in the strip $U = \mathbb{R}^{n-1} \times (0, L)$, where $L > 0$.

   *Hint: These inequalities are called **Poincaré inequalities**. Note that no constant exists for $\mathbb{R}^n$. Compare also to the Sobolev inequalities $\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})}$ and $\|u\|_{L^2(\mathbb{R}^2)} \leq \|Du\|_{L^1(\mathbb{R}^2)}$ for C_c^∞ functions found on the previous two sheets.*