Functional Analysis

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Problem Sheet 5.

Due 4.12.2015.

Problem 1. (Sobolev inequalities) (4+5+1 Points)

a) Let $1 \leq p < \infty$. Show that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.

b) Let $u \in C_c^\infty(\mathbb{R}^2)$. Show that $\|u\|_{L^2} \leq \|Du\|_{L^1}$.
Hint: Write $u^2(x,y) = \left(\int_{-\infty}^x \partial_1 u(t,y)\,dt\right)\left(\int_{-\infty}^y \partial_2 u(x,s)\,ds\right)$.

c) Show that there is no constant $C > 0$ such that $\|u\|_{L^1} \leq C\,\|Du\|_{L^1}$ for all $u \in C_c^\infty(\mathbb{R}^2)$.

Problem 2. (Reproducing kernels) (3+2+5 Points)

Let $\Omega \neq \emptyset$ be a set. Suppose $H \subset \{f : \Omega \to \mathbb{R}\}$ is a real Hilbert space of functions $\Omega \to \mathbb{R}$ with inner product $(\cdot,\cdot)$. We call a function $K : \Omega \times \Omega \to \mathbb{R}$ a reproducing kernel for H if

i) $K(x,\cdot) \in H$ for all $x \in \Omega$, and

ii) $f(x) = (f, K(x,\cdot))$ for all $f \in H$ and all $x \in \Omega$.

a) Prove that if a reproducing kernel for H exists, then it is unique.

b) Prove that if a reproducing kernel for H exists, then, for every $x \in \Omega$, the evaluation functional $\delta_x : H \to \mathbb{R}, f \mapsto f(x)$ is a bounded linear functional.

c) Consider $\Omega = \mathbb{R}$ and set $K(x,y) \coloneqq \frac{1}{2}e^{-|x-y|}$. Prove that K is a reproducing kernel for $W^{1,2}(\mathbb{R})$. Precisely, prove i) and prove for any $f \in W^{1,2}(\mathbb{R})$ that its contiunuous representative $\bar{f}$ satisfies for all $x \in \mathbb{R}$

$$\bar{f}(x) = (\bar{f}, K(x,\cdot))_{W^{1,2}(\mathbb{R})} = \int_{\mathbb{R}} \bar{f}(y)K(x,y)\mathrm{d}y + \int_{\mathbb{R}} \bar{f}'(y)\partial_2 K(x,y)\mathrm{d}y.$$

Hint: Split the second integral and integrate by parts. Recall that $C_c^\infty(\mathbb{R})$ *is dense in* $W^{1,2}(\mathbb{R})$.

Problem 3. (Difference quotients) (4+4+2 Points)

Suppose $U \subset \mathbb{R}^n$ is open and bounded. For i=1,…,n and $h > 0$ define $U_{i,h} = \{x \in U : x + te_i \in U \text{ for all } 0 < t \leq h\}$, and let $1 \leq p < \infty$.

a) Prove that for all $u \in W^{1,p}(U)$, every $i = 1, \dots, n$ and for all $h > 0$

$$\int_{U_{i,h}} \frac{|u(x + he_i) - u(x)|^p}{h^p} \mathrm{d}x \leq \int_U \left|\frac{\partial u}{\partial x_i}\right|^p \mathrm{d}x$$

Hint: For smooth functions use the fundamental theorem of calculus.

b) Suppose $u \in L^p(U)$ satisfies for all $h > 0$

$$\left(\int_{U_{i,h}} \frac{|u(x + he_i) - u(x)|^p}{h^p} \mathrm{d}x\right)^{\frac{1}{p}} \leq C < \infty, \ i = 1, \dots, n.$$

Let $k \mapsto \varphi_k$ be a sequence of mollifiers with $\operatorname{supp} \varphi_k \subset B(0, \frac{1}{k})$, and set $u_k := u * \varphi_k$. Prove that for any $U' \subset U$ open with $\bar{U}' \subset U \setminus \overline{B(\partial U, \frac{1}{k})}$ and all $i = 1, \dots, n$

$$\left(\int_{U'} \left|\frac{\partial u_k}{\partial x_i}\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq C. \tag{1}$$

Hint: write out the convolution explicitly to show

$$\left(\int_{U'_{i,h}} |u_k(x + he_i) - u_k(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_{U_{i,h}} |u(x + he_i) - u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

c) We will see later that (1) implies that if $1 < p < \infty$ there exists a subsequence $l \mapsto u_{k_l}$ and functions $v_i \in L^p(U')$ such that

$$\lim_{l \to \infty} \int_{U'} \frac{\partial}{\partial x_i} u_{k_l}(x) \xi(x) \, dx = \int_{U'} v_i(x) \xi(x) \, dx \text{ for all } \xi \in L^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Show that this implies that $u \in W^{1,p}(U')$.

Problem 4. (Minimization) (5+5+5$*$ Points)

Decide whether the following problems have a minimizer:

a) Let $g \in C^0([0, 1])$. Minimize $\|f - g\|_{L^\infty([0,1])}$ among all $f \in L^\infty([0, 1])$ with $\int_0^1 f \, dx = 0$.
Hint: Consider $\left|\int_0^1 (g - f) \, dx\right|$.

b) Minimize $\int_0^1 (u(x))^2 + ((u'(x))^2 - 1)^2 \, dx$ in $W^{1,4}([0, 1])$.
Hint: Sketch the function $z \mapsto (z^2 - 1)^2$.

c$*$) Minimize $\int_{-1}^1 (u'(x))^2 \, dx$ among all $u \in W^{1,2}([-1, 1])$ with $u(-1) = u(1) = 0$ and

$$u(x) \geq \max(1 - 2\,|x|\,, 0)$$

for all $x \in [0, 1]$. This is called an **obstacle problem**.

Hint: The pointwise conditions on u are well-defined for the continuous representative (see Problem 4 on Sheet 4).