Functional Analysis

WS 2015/2016
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Problem Sheet 4.

Due 27.11.2015.

Problem 1. (L^p norms) (5+5 Points)

Let $\Omega \subset \mathbb{R}^n$ is such that $0 < |\Omega| = \int_\Omega 1 \mathrm{d}x < \infty$. Let $u : \Omega \to \mathbb{R}$ be measurable. Define $\Phi_u : [1, \infty) \to [0, \infty]$ by

$$\Phi_u(p) = \left(\frac{1}{|\Omega|}\right)^{\frac{1}{p}} \|u\|_{L^p(\Omega)} .$$

a) Prove that Φ_u is nondecreasing (in particular for $p < q$ if $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$).

b) Prove that $u \in L^\infty(\Omega)$ if and only if the limit $\lim_{p\to\infty} \Phi_u(p)$ is finite and that in this case $\lim_{p\to\infty} \Phi_u(p) = \|u\|_{L^\infty(\Omega)}$.

Problem 2. (The Fourier transform) (5+5+5∗+5∗ Points)

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n, \mathbb{C})$ as the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined as

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x}\, dx.$$

Remember that for $f, g \in L^2(\mathbb{R}^n, \mathbb{C})$ the Fourier transforms $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n, \mathbb{C})$ are well-defined and $(f, g)_{L^2} = (\hat{f}, \hat{g})_{L^2}$. This is known as Plancharel's identity.

a) Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Show that $\widehat{\partial_j \phi}(k) = ik_j \widehat{\phi}(k)$ for $j = 1, \dots, n$.

b) Let $f \in W^{1,2}(\mathbb{R}^n)$. Show that $\widehat{\partial_j f}(k) = ik_j \widehat{f}(k)$, where $\partial_j f \in L^2(\mathbb{R}^n)$ is the weak derivative of f with respect to x_j.

c∗) Let $f \in L^2(\mathbb{R}^n)$. Show that f is in $W^{1,2}(\mathbb{R}^n)$ if and only if the map $k \mapsto |k| \hat{f}(k)$ is in $L^2(\mathbb{R}^n)$, with

$$\|f\|_{W^{1,2}}^2 = \int_{\mathbb{R}^n} (1 + |k|^2) \left|\hat{f}(k)\right|^2 \mathrm{d}k.$$

Conclude for $f \in W^{m,2}(\mathbb{R}^n)$ that

$$\|f\|_{W^{m,2}}^2 = \int_{\mathbb{R}^n} \left(\sum_{l=0}^{m} |k|^{2l}\right) \left|\hat{f}(k)\right|^2 \mathrm{d}k.$$

d∗) Prove that there is a constant $C > 0$ such that for all $f \in W^{2,2}(\mathbb{R}^n)$

$$\|f\|_{W^{1,2}}^2 \le C \|f\|_{L^2} \|f\|_{W^{2,2}} .$$

Problem 3. (Sobolev functions) (6+4 Points)

a) Let $n \in \mathbb{N}$. Find all pairs $\alpha \in \mathbb{R}$, $p \in [1, \infty]$, such that the function $x \in \mathbb{R}^n \mapsto \|x\|^\alpha$ is in $W^{1,p}(B(0,1))$.

   Hint: To show that the piecewise derivative of an almost everywhere differentiable function is indeed its weak derivative, cut out a small ball around the singularity and integrate by parts in the rest of the domain.

b) Show that the Heaviside step function $H : \mathbb{R} \to \mathbb{R}$,

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   is not in $W^{1,1}((-1,1))$ even though it has derivative 0 almost everywhere.

Problem 4. (Absolutely continuous) (5+2+3 Points)
Let $g \in L^1((0,1))$. Define $f(x) = \int_0^x g(t)\,dt$.

a) Show that $f \in W^{1,1}((0,1))$ with weak derivative $f' = g$.

b) Let $h \in W^{1,1}((0,1))$ with weak derivative $h' \in L^1((0,1))$. Show that $h(x) = \int_0^x h'(t)\,dt + \text{const}$ almost everywhere. In particular show that h is continuous.

c) Let $h \in W^{1,p}((0,1))$ with $p \in (1, \infty]$. Show that h has a representative in $C^{0,1-1/p}([0,1])$.

Hint: For (iii) use Hölder's inequality.

The student council of mathematics will organize the math party on 26.11. in N8schicht. The presale will be held on Mon 23.11., Tue 24.11. and Wed 25.11. in front of the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de