Functional Analysis

WS 2015/2016
Prof. Dr. M. Disertori
P. Gladbach; R. Schubert

universitätbonn

Problem Sheet 2.

Due 13.11.2015.

Problem 1. (Convexity) (3+5+2+5∗ Points)

a) Let $(X, \|\cdot\|)$ be a normed space. Show that the closed unit ball $\overline{B(0,1)}$ is convex.

b) Let $(X, (\cdot,\cdot))$ be a pre-Hilbert space. Show that the closed unit ball is *strictly convex*, i.e. that for $x, y \in \overline{B(0,1)}$, $x \neq y$, and $t \in (0,1)$ we have $(1-t)x + ty \in B(0,1)$, i.e. the interior of $\overline{B(0,1)}$.

Hint: Compute the second derivative of the function $t \mapsto \|(1-t)x + ty\|^2$.

c) Sketch the unit ball $\overline{B(0,1)}$ in $(\mathbb{R}^2, \|\cdot\|_p)$ for $p = 1, 3/2, 2, 3, \infty$. What about $p = 1/2$?

d∗) Show that the closed unit ball in $(\mathbb{R}^n, \|\cdot\|_p)$ is strictly convex for $p \in (1, \infty)$.

Problem 2. (The spaces l_p) (5+3+2 Points)

Define the norm $\|\cdot\|_p : \mathbb{R}^{\mathbb{N}} \to [0,\infty]$ for $p \in [1,\infty)$ as $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$ and the space $l_p = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_p < \infty\}$.

a) Show that $(l_p, \|\cdot\|_p)$ is a Banach space.

b) Find all pairs $p, q \in [1,\infty)$ such that $\mathrm{id} : l_p \to l_q$ is a bounded linear operator.

c) Find an injective, non-surjective bounded linear operator $T : l_p \to l_p$.

Hint: You may use that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ for all $n \in \mathbb{N}$.
Let X, Y be normed spaces. A linear operator $A : X \to Y$ is bounded if there exists $C \in (0,\infty)$ such that $\|Ax\|_Y \leq C \|x\|_X$ for all $x \in X$.

Problem 3. (Completeness) (4+6 Points)

a) Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x,y) = |\arctan x - \arctan y|$. Prove that d is a metric on $\mathbb{R}$, and that the induced topology agrees with the standard topology on $\mathbb{R}$. Show further that $(\mathbb{R}, d)$ is not complete, i.e. there is a Cauchy sequence $x : \mathbb{N} \to \mathbb{R}$ which does not converge with respect to d.
Hint: You may use that (with respect to the standard topology) $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ *is a homeomorphism and* $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$.

b) Are the following normed spaces complete? (If not, give a counterexample.)

   i) l_1 equipped with the l_2 norm.
   ii) $C_c^0(\mathbb{R}^n)$ equipped with the sup norm $\|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|$.
   iii) $C^0[0,1]$ equipped with the L^1 norm $\|f\|_{L^1} = \int_0^1 |f(x)|\,\mathrm{d}x$.

Problem 4. (Parallelogram identity) (10 Points)

Let $(V, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space. Prove that the norm $\|\cdot\|$ is given by a scalar product $(\cdot, \cdot)$, i.e. $\|x\| = \sqrt{(x, x)}$ if and only if the parallelogram identity

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

holds for all $x, y \in V$.
Hint: For the 'if' part construct a candidate for the scalar product by looking at $\|x+y\|^2 = (x+y, x+y) = \dots$.