Nonlinear Partial Differential Equations I
Winter term 2016/2017
Problem Sheet 13 (optional)

University of Bonn
Prof. Dr. M. Disertori
Dr. M. Bonacini

Problem 1 (Lower semicontinuity).
Let X be a topological space and let $F : X \to \mathbb{R}$.

a) Prove that the following properties are equivalent:

   (i) F is (sequentially) lower semicontinuous;

   (ii) for every $t \in \mathbb{R}$ the set $\{x \in X \,:\, F(x) \leq t\}$ is (sequentially) closed.

b) Assume that X is a Banach space, and let $F : X \to \mathbb{R}$ be convex. Show that F is (sequentially) lower semicontinuous with respect to the *strong* topology if and only if it is (sequentially) lower semicontinuous with respect to the *weak* topology.
Recall that a convex set is closed in the strong topology if and only if it is closed in the weak topology: see, for instance, Brezis, Functional Analysis.

Problem 2 (Direct method of the Calculus of Variations).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $1 < p < \infty$. Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following assumptions:

a) for every $\xi \in \mathbb{R}^n$ the map $x \mapsto f(x, \xi)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $\xi \mapsto f(x, \xi)$ is continuous and convex on $\mathbb{R}^n$;

c) $f(x, \xi) \geq a(x) + b|\xi|^p$ for some $a \in L^1(\Omega)$ and $b > 0$.

Define the functional

$$F(u) := \int_\Omega f(x, \nabla u(x))\, \mathrm{d}x\,, \qquad u \in W_0^{1,p}(\Omega).$$

Prove that:

a) F is sequentially lower semicontinuous with respect to the strong topology of $W_0^{1,p}(\Omega)$.

b) There is a solution to the minimum problem

$$\min_{u \in W_0^{1,p}(\Omega)} F(u)\,.$$

Hint: use the result in Problem 1.

Please turn over.

Problem 3 (Nonexistence of minimizers: lack of compactness).
Let $X = \{u \in W^{1,1}(-1,1) \,:\, u(-1) = -1,\, u(1) = 1\}$, and let

$$F(u) = \int_{-1}^{1} \big(1 + |x|\big)|u'(x)| \,\mathrm{d}x\,.$$

Show that $\inf_{u\in X} F(u) = 2$ and that F has no minimizer in X.

Problem 4 (Nonexistence of minimizers: lack of convexity).
Let

$$F(u) = \int_{0}^{1} \Big[u(x)^2 + \big(u'(x)^2 - 1\big)^2\Big] \,\mathrm{d}x\,.$$

Show that $\inf_{u\in H_0^1(0,1)} F(u) = 0$ and that F has no minimizer in $H_0^1(0,1)$.

Problem 5 (Necessity of convexity for lower semicontinuity).
In this exercise we show the construction used to generalize to dimensions $n > 1$ the proof of the theorem seen in the lecture on Thursday.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function such that for every $R > 0$ there exists $g_R \in L^1(\Omega)$ with $|f(x,p)| \le g_R(x)$ for almost every $x \in \Omega$ and for every $p \in \mathbb{R}^n$. Assume that the functional

$$F(u) = \int_{\Omega} f(x, \nabla u(x)) \,\mathrm{d}x\,, \qquad u \in W^{1,\infty}(\Omega),$$

is sequentially lower semicontinuous with respect to the weak*-convergence in $W^{1,\infty}(\Omega)$. The aim is to show that the map $p \mapsto f(x,p)$ is convex for almost every $x \in \Omega$.

Fix $p_1, p_2 \in \mathbb{R}^n$ and $\lambda \in (0,1)$, and let $p = \lambda p_1 + (1-\lambda)p_2$. Let $e := \frac{p_2 - p_1}{|p_2 - p_1|}$ and decompose the three vectors p, p_1, p_2 in their components parallel and orthogonal to e:

$$p_1 = p_1^{\|} e + p_1^{\perp}, \quad p_2 = p_2^{\|} e + p_2^{\perp}, \quad p = p^{\|} e + p^{\perp},$$

(notice that $p_1^{\perp} = p_2^{\perp} = p^{\perp}$). Let $u_k : \mathbb{R} \to \mathbb{R}$ be the functions defined in the lecture, which satisfy the property

$$u_k' = \begin{cases} p_1^{\|} & \text{on } A_k = (\frac{m}{k}, \frac{m+\lambda}{k}),\ m \in \mathbb{Z}, \\ p_2^{\|} & \text{on } \mathbb{R}\backslash A_k, \end{cases} \qquad\qquad u_k' \rightharpoonup \lambda p_1^{\|} + (1-\lambda) p_2^{\|} \quad \text{in w*-}L^{\infty}.$$

Finally, define $\tilde{u}_k(x) = u_k(x \cdot e) + x \cdot p^{\perp}$.

Show that $\tilde{u}_k \rightharpoonup u_p$ weakly* in $W^{1,\infty}$, where $u_p(x) = x \cdot p$, and deduce that

$$\int_{\Omega} f(x,p)\,\mathrm{d}x \le \lambda \int_{\Omega} f(x,p_1)\,\mathrm{d}x + (1-\lambda)\int_{\Omega} f(x,p_2)\,\mathrm{d}x\,,$$

for every $p = \lambda p_1 + (1-\lambda)p_2$, with $p_1, p_2 \in \mathbb{R}^n$, $\lambda \in (0,1)$. From this point the proof of the convexity of $f(x,\cdot)$ follows exactly as in the one-dimensional case.
