Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 12 (optional) Dr. M. Bonacini

Problem 1 (Poincaré inequality in an interval).
Let $I = (0,1) \subset \mathbb{R}$, let $\lambda \geq \pi$ and let $L(s,p) = \frac{1}{2}(p^2 - \lambda^2 s^2)$. Consider the integral functional

$$I_\lambda(u) = \int_0^1 L(u(x), u'(x))\,\mathrm{d}x = \frac{1}{2}\int_0^1 |u'|^2\,\mathrm{d}x - \frac{\lambda^2}{2}\int_0^1 |u|^2\,\mathrm{d}x\,, \qquad \text{for } u \in H_0^1(0,1).$$

a) For $\lambda = \pi$, find a family of solutions $u_\alpha \in H_0^1(0,1)$, depending on a real parameter $\alpha \in \mathbb{R}$, to the Euler-Lagrange equation associated with the functional I_λ, such that $I_\lambda(u_\alpha) = 0$.

b) For $\lambda > \pi$, show that $\inf_{u\in H_0^1(0,1)} I_\lambda(u) = -\infty$, and hence the functional I_λ has no minimizer in $H_0^1(0,1)$. (It is also worth to remark that for $0 \leq \lambda \leq \pi$ one has $\inf_{u\in H_0^1(\Omega)} I_\lambda(u) = 0$: hence $\lambda = \pi$ is the best constant for the Poincaré inequality in an interval).

c) However, show that the Euler-Lagrange equation has always a solution, and that for some values of $\lambda > \pi$ there are infinitely many solutions in $H_0^1(0,1)$. (Notice that the map $p \mapsto f(s,p)$ is convex, but $(s,p) \mapsto f(s,p)$ is not, and in this case there are solutions to the Euler-Lagrange equation which are not minimizers).

Problem 2 (Minimal graphs of revolution).
Let $u \in C^1([-1,1])$ such that $u > 0$, $u(-1) = a$, $u(1) = b$ for given $a, b > 0$. The area of the surface obtained by rotating the graph of u about the x-axis is

$$A(u) = 2\pi \int_{-1}^1 u(x)\sqrt{1 + |u'(x)|^2}\,\mathrm{d}x\,.$$

a) Derive the Euler-Lagrange equation of A.

b) Show that, if u is a (sufficiently regular) minimizer of A in the class of functions with the same boundary values, then

$$u^2 = c^2(1 + |u'|^2) \tag{1}$$

for some constant $c \in \mathbb{R}$.

c) Solve the equation (1). Is the solution always a minimizer of the area among surfaces of revolution with the same boundary values? Is there always a minimizer among smooth graphs with prescribed boundary values? Think about the case $a = b$.
Hint: look for a solution in the form $u(x) = c\cosh v(x)$.

Problem 3 (Minimal surfaces).
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Show that if the area of the graph of u, defined by

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, \mathrm{d}x\,,$$

is minimal with respect to every $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $v = u$ on $\partial\Omega$, then the graph of u has mean curvature constantly equal to 0, that is

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \qquad \text{in } \Omega.$$

Problem 4 (Variational formulation of a nonlinear equation).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function:

a) for every $s \in \mathbb{R}$ the map $x \mapsto f(x, s)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $s \mapsto f(x, s)$ is continuous on $\mathbb{R}$.

Assume also that f satisfies the growth condition

$$|f(x, s)| \le a(x) + b|s|$$

for some $a \in L^2(\Omega)$ and $b > 0$. Find an integral functional $I : H^1_0(\Omega) \to \mathbb{R}$ such that every minimizer of I in $H^1_0(\Omega)$ is a weak solution to the boundary value problem

$$\begin{cases} -\Delta u = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Problem 5 (Euler-Lagrange equations).
Find a Lagrangian $L = L(x, s, p)$ such that the PDE

$$-\Delta u + D\phi \cdot Du = f \qquad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to the functional $I(u) = \int_\Omega L(x, u(x), Du(x))\, \mathrm{d}x$. Here ϕ and f are given smooth functions.
Hint: look for a Lagrangian with an exponential term.
