Nonlinear Partial Differential Equations I University of Bonn
Winter term 2016/2017 Prof. Dr. M. Disertori
Problem Sheet 11 (due Monday 23.01.2017) Dr. M. Bonacini

Problem 1 (Nonlinearities and weak convergence, 6 points).
The aim of the exercise is to show that the weak convergence of a sequence $f_n \rightharpoonup f$ in L^2 does **not** imply $a(f_n) \rightharpoonup a(f)$ for any **nonlinear**, real-valued function a.

a) *(Weak convergence of highly-oscillating functions)* Let $u : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function in $L^\infty(\mathbb{R})$, and define $u_n(x) := u(nx)$ for $n \in \mathbb{N}$. Show that, as $n \to \infty$,

$$u_n \rightharpoonup m = \int_{(0,1)} u(x)\,\mathrm{d}x \qquad \text{weakly in } L^2(A), \text{ for every open, bounded set } A \subset \mathbb{R}.$$

Hint: by considering the functions $U(x) = \int_0^x (u(t) - m)\,\mathrm{d}t$, $U_n(x) = U(nx)$, and integrating by parts, show that $\lim_{n\to\infty} \int_{\mathbb{R}} (u_n(x) - m)\varphi(x)\,\mathrm{d}x = 0$ for every $\varphi \in C^1_{\mathrm{c}}(\mathbb{R})$.

b) Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $a(f_n) \rightharpoonup a(f)$ weakly in $L^2(0,1)$ whenever $f_n \rightharpoonup f$ weakly in $L^2(0,1)$. Prove that a is affine:

$$a(z) = \alpha z + \beta\,,$$

for some constants α, β.
Hint: suppose by contradiction that there exist values $z_1, z_2 \in \mathbb{R}$ and $\lambda \in (0,1)$ such that $a(\lambda z_1 + (1-\lambda)z_2) \neq \lambda a(z_1) + (1-\lambda)a(z_2)$. Then construct a suitable weakly convergent sequence f_n, and use the result in a).

c) Find a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that for every $p \in \mathbb{R}$ there is a sequence $f_n \in L^\infty(0,1)$ such that $f_n \rightharpoonup 0$ and $a(f_n) \rightharpoonup p$ weakly in $L^2(0,1)$.

Problem 2 (p-Laplacian, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $p \in (1,\infty)$. Consider the vector field $a : \mathbb{R}^n \to \mathbb{R}^n$ given by $a(\xi) := |\xi|^{p-2}\xi$, and the associated operator

$$A : W^{1,p}_0(\Omega) \to (W^{1,p}_0(\Omega))'$$

defined by $A(u) := -\mathrm{div}\big(a(\nabla u)\big)$, or equivalently

$$A(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x \qquad \text{for every } u, v \in W^{1,p}_0(\Omega).$$

Show that:

a) A is well-defined and maps bounded sets into bounded sets;

b) A is strictly monotone: $(A(u) - A(v))(u - v) > 0$ for every $u, v \in W^{1,p}_0(\Omega)$, $u \neq v$;

c) A is coercive on $W^{1,p}_0(\Omega)$.

Problem 3 (Mean curvature of a graph, 6 points).
For a function $u : \Omega \to \mathbb{R}$ of class C^2 in an open set $\Omega \subset \mathbb{R}^n$, the mean curvature H of the graph of u has the expression

$$nH = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \qquad \text{in } \Omega. \tag{1}$$

Suppose that $\Omega \subset B(0,R)$ for some $R > 0$, and define $v_R : \Omega \to \mathbb{R}$ by $v_R(x) = \sqrt{R^2 - |x|^2}$.

a) Show that the graph of v_R has mean curvature equal to $-\frac{1}{R}$.

b) Let $H : \Omega \to \mathbb{R}$ with $-\frac{1}{R} < H < 0$, and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = nH \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Show that $0 < u < v_R$ in Ω.
Hint: to show that $u < v_R$, notice that the equation for u is equivalent to

$$\frac{1}{\sqrt{1+|\nabla u|^2}}\left(-\Delta u + \sum_{i,j=1}^{n} \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\partial_i\partial_j u\right) = -nH\,.$$

Assume that $u - v_R$ has an interior maximum at $x_0 \in \Omega$, and derive a contradiction. To show that $u > 0$ argue similarly, or apply the strong maximum principle for linear PDEs.

c) Show that the vector field $a(p) = \frac{p}{\sqrt{1+|p|^2}}$ is strictly monotone, that is for every $p, q \in \mathbb{R}^n$ $(a(p) - a(q)) \cdot (p - q) > 0$. Is it possible to apply the general theory seen in the lectures to obtain the existence of a weak solution u to (1), for a given $H \in L^2(\Omega)$?

Total: 16 points